The Dold-Thom theorem

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Abstract — The main aim of this project consists in giving an unusual approach of singular homology theory through the Dold-Thom theorem, which states in particular the following interesting result in algebraic topology:

$$\tilde{H}_n(X;\,\mathbb{Z}) = \pi_{n+1}\left(SP(\Sigma\tilde{X})\right),$$

for any integer $n \geq 0$ and any pointed space X (see the theorem 3.2.11). In other words, we will be led throughout these very few pages toward a relation at the crossing between homology and homotopy theories. After considering elementary notions in category theory, we will then sprinkle some pushouts and exact sequences on our reasonings in order to work on homotopy and homology theories. Furthermore, the suspension and the reduced cone will reveal themselves as very useful tools, without taking cofibrations nor CW-complexes into account. Thus, we will adopt a homotopic point of view as developed in the reference [AGP02], instead of the common geometric one as in [Hat01, page 108], and we will deeply explore various properties of some axioms and categories.

Keywords — Category, Pushout, Homotopy, Exact Sequence, Homology, Suspension, CW-complex.

Contents

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Introduction

To classify in various ways different objects has always been an important goal for mathematicians throughout generations. For some reason, it is more particularly interesting to distinguish objects according to the number of wholes one can enumerate: this is homotopy theory. In this sense, we often say that a doughnut could be seen as a coffee cup since both of them "contain" one single whole. Nonetheless, it remains quite complicate to clarify the definition of whole, as far as it is actually the absence of "matter".

This is one of the reasons why the fundamental group π_1 was introduced: given an object with holes, we try to coil some loops around the holes, and then we classify together objects in which loops coil in a similar way around holes.

As always, once they clearly understood this notion, mathematicians then wanted to find generalizations to higher dimensions: this is why they introduced homotopy groups π_n for integers $n \geq 0$. Now, instead of evaluating wholes with loops, we work with multidimensional spheres. However, at this point, computations become so much intricate that they chose to consider another theory: homology.

All of sudden, computations are easier; we will even see some particular cases in this project. Indeed, the cunning results from transforming in a certain way topological space into abelian groups. Nevertheless, the other side of the coin is that homology is more delicate to introduce...

Now, we would like to find links between homotopy and homology theories. To this end, the mathematicians Albrecht Dold and René Thom proved in the 1950's a theorem that provides such a relation.

— PART 1 —

Axiomatic homology

In this part, the goal is to introduce the axioms developed by Samuel Eilenberg and Norman E. Steenrod in mid XX^{th} century (see the reference [ES45, pages 117-120]). To do so, we will need to define some concepts taken from category theory, such as functors, pushouts or also natural transformations. We will moreover consider a few kinds of equivalences, namely not only isomorphisms $\cong$, but also homotopy equivalences $\simeq$ and weak equivalences $\sim$. Finally, suspensions and reduced cones will be very useful tools in order to compute the homology of the n-spheres.

I. Category theory

1. Generalities

Category theory will be the solid foundation of this project. Most of what we will see is provided by the reference [Bor94]. This theory actually gives a generalization of the organisation we can find between mathematical objects and morphisms.

Definitions 1.1.1. *A **category** $\mathcal{C}$ is the following data:*

(i) *A collection $Ob(\mathcal{C})$, whose elements are called **objects**.*

(ii) *For each pair of objects $X, Y \in Ob(\mathcal{C})$, we define a set $\mathcal{C}(X, Y)$ whose elements $f \in \mathcal{C}(X, Y)$ are called **morphisms** from X to Y and written $f : X \to Y$.*

(iii) *A **composition**:*

$$\forall X, Y, Z \in Ob(\mathcal{C}), \qquad \begin{cases} \mathcal{C}(X, Y) \times \mathcal{C}(Y, Z) & \to & \mathcal{C}(X, Z), \\ (f, g) & \mapsto & g \circ f, \end{cases}$$

subjects to :

- ***Associativity**: for any objects $X, Y, Z, W \in Ob(\mathcal{C})$, we have:*

$$h \circ (g \circ f) = (h \circ g) \circ f,$$

where $f \in \mathcal{C}(X, Y)$, $g \in \mathcal{C}(Y, Z)$ and $h \in \mathcal{C}(Z, W)$.

- ***Identity**: for all object $X \in Ob(\mathcal{C})$, there is a morphism $id_X \in \mathcal{C}(X, X)$ such that:*

$$f \circ id_X = f \qquad and \qquad id_X \circ f' = f',$$

where $f \in \mathcal{C}(X, Y)$, $f' \in \mathcal{C}(Y, X)$ and $Y \in Ob(\mathcal{C})$.

Remark 1.1.2. The identity id_X is the only morphism from X to X which plays the role of an identity for the composition law. Indeed, if $1_X \in \mathcal{C}(X, X)$ is another such morphism, then:

$$id_X = id_X \circ 1_X = 1_X.$$

Notation 1.1.3. *We denote $Mor(\mathcal{C})$ the set of all the morphisms of $\mathcal{C}$:*

$$Mor(\mathcal{C}) := \{morphisms\ of\ \mathcal{C}\} = \bigcup_{X, Y \in Ob(\mathcal{C})} \mathcal{C}(X, Y).$$

Definition 1.1.4. *A morphism $f \in \mathcal{C}(X, Y)$ is an **isomorphism** of $\mathcal{C}$ if there exists $g \in \mathcal{C}(Y, X)$ such that $g \circ f = Id_X$ and $f \circ g = Id_Y$. In such a case, we note $X \cong Y$. Moreover, we denote $Iso(\mathcal{C}) \subseteq Mor(\mathcal{C})$ the set of isomorphisms of $\mathcal{C}$.*

Examples. Here are two examples of categories. You can find the table 1 in appendix that inventories a complete list of the categories used in this project.

(I) $\mathcal{C} := \mathsf{Top}$ (category of topological spaces),
$Ob(\mathsf{Top}) :=$ {topological spaces},
$Mor(\mathsf{Top}) :=$ {continuous maps} (where composition is the usual),
$Iso(\mathsf{Top}) =$ {homeomorphisms}.
Note that we can similarly introduce Top_*, taking pointed spaces (X, x_0) and pointed continuous maps $f : (X, x_0) \to (Y, y_0)$ (*i.e.* $f : X \to Y$ is continuous and $f(x_0) = y_0$).

(II) $\mathcal{C} := \mathsf{Gr}$ (category of groups),
$Ob(\mathsf{Gr}) :=$ {groups},
$Mor(\mathsf{Gr}) :=$ {homomorphims of groups},
$Iso(\mathsf{Gr}) =$ {isomorphisms of groups}.
In the same way, we can introduce the category Ab of the abelian groups.

Definition 1.1.5. *A **functor** $F : \mathcal{C} \to \mathcal{D}$ between categories $\mathcal{C}$ and $\mathcal{D}$ is given by two maps of set:*

$$F : Ob(\mathcal{C}) \to Ob(\mathcal{D}) \qquad \text{and} \qquad F : \mathcal{C}(X, Y) \to \mathcal{D}(F(X), F(Y)),$$

for all $X, Y \in Ob(\mathcal{C})$, such that:

(i) *F preserves composition: for all $g \in \mathcal{C}(Y, Z)$ and $f \in \mathcal{C}(X, Y)$, we have $F(g \circ_{\mathcal{C}} f) = F(g) \circ_{\mathcal{D}} F(f)$, where $X, Y, Z \in Ob(\mathcal{C})$.*

(ii) *F preserves identities: $F(id_X) = id_{F(X)}$, where $X \in Ob(\mathcal{C})$.*

Remark 1.1.6. Given two functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$, we can construct a third functor: $G \circ F : \mathcal{C} \to \mathcal{E}$, where $\circ$ is the usual composition in Set.

One can define a very similar notion: contravariant functors. Note that contravariant functors are not functors, but they will verify alike properties.

Definition 1.1.7. *Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A **contravariant functor** $F : \mathcal{C} \to \mathcal{D}$ is a mapping that associates to each object X in $\mathcal{C}$ an object $F(X)$ in $\mathcal{D}$, and to each morphism $f : X \to Y$ of $\mathcal{C}$ a morphism $f^* := F(f) : F(Y) \to F(X)$ in $\mathcal{D}$, such that:*

(i) *$F(f \circ_{\mathcal{C}} g) = F(g) \circ_{\mathcal{D}} F(f)$ for all $f : Y \to Z$ and $g : X \to Y$ in $\mathcal{C}$,*

(ii) *$F(id_X) = id_{F(X)}$ for any $X \in Ob(\mathcal{C})$.*

Proposition 1.1.8. Given a functor $F : \mathcal{C} \to \mathcal{D}$, if $f \in Mor(\mathcal{C})$ is an isomorphism, then so is $F(f)$ in $\mathcal{D}$. In other words, if we have $A \cong B$ for two objects A and B of $\mathcal{C}$, then we have $F(A) \cong F(B)$ in $\mathcal{D}$.

Proof. Let be X and Y two objects of $\mathcal{C}$, and $f \in Iso(\mathcal{C})$ an isomorphism from X to Y. We can find a morphism $g \in \mathcal{C}(Y, X)$ such that $g \circ f = Id_X$ and $f \circ g = Id_Y$. As F lands in $Mor(\mathcal{D})$, we have that $F(f)$ and $F(g)$ are morphisms of $\mathcal{D}$. Now $F(f)$ is well an isomorphism from $F(X)$ to $F(Y)$ because:

$$F(g) \circ F(f) = F(g \circ f) = F(id_X) = id_{F(X)} \qquad \text{and} \qquad F(f) \circ F(g) = F(f \circ g) = F(id_Y) = id_{F(Y)}. \qquad \square$$

Remark 1.1.9. This is why we can consider two homeomorphic relative space as being the same. This result remains true for the contravariant functors.

Remark 1.1.10. One could wander if it is possible to build the category of the categories, with categories as objects and functors as morphisms. Nonetheless, as detailed in [Bor94, page 6], it would induces a contraction with the definition of set.

Remark 1.1.11. In algebraic topology, what is interesting is to have functors $F : \mathcal{C} \to \mathcal{D}$, where $\mathcal{C}$ is something topological and $\mathcal{D}$ is something algebraic. For instance, the fundamental group $\pi_1 : \mathsf{Top}_* \to \mathsf{Gr}$ is such a functor. Indeed, for any continuous map $f : (X, x_0) \to (Y, f(x_0))$ in Top_*, one can see that:

$$\pi_1(f) = f_* : \begin{cases} \pi_1(X, x_0) & \to & \pi_1(Y, f(x_0)), \\ [c] & \mapsto & [f \circ c], \end{cases}$$

is a group homomorphism. It also satisfies the association $(g \circ f)_* = g_* \circ f_*$, and the identity $(Id_{(X, x_0)})_* = id_{\pi_1(X, x_0)}$. Note that we will see the same result in 1.2.13 as a particular case of a more general result.

Remark 1.1.12. One can find in the literature the notion of **forgetful functors**: they are functors that "lose some data". Here are a few examples:

(I) $U : \mathsf{Top} \to \mathsf{Set}$, $(X, \tau) \mapsto X$, and that maps a continuous map f to the corresponding map f.

(II) $U : \mathsf{Gr} \to \mathsf{Set}$, $(G, +) \mapsto G$, and that maps a group homomorphism f to the corresponding map f.

(III) $U : \mathsf{Top}_* \to \mathsf{Top}$, $(X, x_0, \tau) \mapsto (X, \tau)$, and that maps a continuous pointed map f to the corresponding continuous map f.

Conversely, here are two examples of **free functors** that "gain some data":

(I) $V : \mathsf{Set} \to \mathsf{Ab}$, $X \mapsto \mathbb{Z}\langle X \rangle$.

(II) $V : \mathsf{Set} \to \mathsf{Top}$, $X \mapsto (X, \text{discrete topology})$, where every set is open.

Definition 1.1.13. *A **commutative diagram** in a category $\mathcal{C}$ is a diagram:*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"'] & C \arrow[d, "g"] \\ B \arrow[r, "k"'] & D, \end{tikzcd}$$

such that $g \circ f = k \circ h$, where $A, B, C, D \in Ob(\mathcal{C})$, and $f, g, k, h \in Mor(\mathcal{C})$.

Remark 1.1.14. We can extend this definition to more complex diagram shapes. And more generally, if a diagram "contains" a couple of squares (or any other "elementary diagram shape"), we consider that the diagram commutes *if, and only if,* each single shape is commutative.

2. Some properties and definitions in *Set*, *Top* and *Gr*

Before going any further, we want to recall two general results in Top_* and Gr that we will use many time throughout this project.

Proposition 1.1.15. Universal Property of Quotient Space. Let (X, x_0) and (Y, y_0) be pointed spaces in Top_*. We also consider an equivalence relation $\approx$ on X, and $[-] : (X, x_0) \to (X/\approx, [x_0])$ the canonical projection, where $X/\approx$ is furnished with the quotient space topology. Then, given any continuous map $f : (X, x_0) \to (Y, y_0)$ in Top_* that is $\approx$-invariant (i.e. $x \approx x'$ implies $f(x) = f(x')$), there exists a unique continuous map $\bar{f} : (X/\approx, [x_0]) \to (Y, y_0)$ such that the following diagram commutes:

$$\begin{tikzcd} (X, x_0) \arrow[r, "f"] \arrow[d, "{[-]}"'] & (Y, y_0) \\ (X/\approx, [x_0]) \arrow[ur, dashed, bend right, "\exists! \bar{f}"'] & \end{tikzcd} \tag{1.1}$$

Proof. As f is $\approx$-invariant, one can define the map $\bar{f} : [x] \mapsto f(x)$, and we observe it is the only map that preserves the commutativity of the diagram (1.1). Moreover, it well sends $[x_0]$ to y_0, so $\bar{f}$ is pointed. Now, let us show that $\bar{f}$ is continuous. We consider an open set $U \subset Y$. As f is continuous, its inverse image $f^{-1}(U)$ is open in X. Then, as the topology on $X/\approx$ is the quotient topology, *i.e.* the finest that allows the natural projection $[-]$ to be continuous, we have that $[f^{-1}(U)]$ is also open, that is to say $\bar{f}^{-1}(U)$ is well open in $X/\approx$. $\square$

Proposition 1.1.16. Universal Property of Quotient Group. Let G and H be two groups in Gr. We also consider a normal subgroup $N \lhd G$ of G, and $\pi : G \to G/N$ the canonical projection. Then, given any group homomorphism $\phi : G \to H$ in Gr which verifies $N \subseteq \ker(\phi)$, there exists a unique group homomorphism $\bar{\phi} : G/N \to H$ such that the following diagram commutes:

$$\begin{tikzcd} G \arrow[r, "\phi"] \arrow[d, "\pi"'] & H \\ G/N \arrow[ur, dashed, bend right, "\exists!\bar{\phi}."'] & \end{tikzcd} \tag{1.2}$$

Proof. First, notice that the quotient $G/N := \{gN \mid g \in G\}$ is well a group since N is normal: the operation is $(gN) \cdot (hN) := (gh)N$ for any gN and hN in G/N. Indeed, this operation is well defined because it does not depend on the choice of representatives g and h: for any $gN = aN$ and $hN = bN$ in G/N,

$$(gh)N \;=\; g(hN) \;=\; g(bN) \;=\; g(Nb) \;=\; (gN)b \;=\; (aN)b \;=\; (Na)b \;=\; N(ab) \;=\; (ab)N.$$

If $N = \{0\}$, then G/N is isomorphic to G and the result is trivial. So we can assume *w.l.o.g.* that $N \neq \{0\}$, and thus we can find an $n \in N$ such that $n \neq 0$. We want to show that the map $\bar{\phi} : gN \mapsto \phi(g)$ is a well-defined group homomorphism. If $gN = hN$, then as $N \subseteq \ker(\phi)$, we have:

$$\bar{\phi}(gN) \;=\; \phi(g) \;=\; \phi(g\,n\,n^{-1}) \;=\; \phi(g)\underbrace{\phi(n)}_{=0}\phi(n^{-1}) \;=\; 0 \;=\; \phi(h)\underbrace{\phi(n)}_{=0}\phi(n^{-1}) \;=\; \phi(h) \;=\; \bar{\phi}(hN),$$

which means that $\bar{\phi}$ is well defined. And we can see it is also a group homomorphism since:

$$\forall gN,\, hN \in G/N, \qquad \bar{\phi}((gN)\cdot(hN)) \;=\; \bar{\phi}((gh)N) \;=\; \phi(gh) \;=\; \phi(g)\,\phi(h) \;=\; \bar{\phi}(gN)\,\bar{\phi}(hN).$$

Moreover, by construction, one can see that $\bar{\phi}$ is the only group homomorphism that preserves the commutativity of the diagram (1.2). ☐

Definitions 1.1.17. *Let $J \in Ob(\mathsf{Set})$ be an index set, and $X \in Ob(\mathsf{Set})$ another set. A J-**tuple** of elements of X is a map $x \in \mathsf{Set}(J, X)$ that we often denote $(x_j)_{j\in J}$, where the image of a $j \in J$ is noted x_j and called the j-**th coordonate** of x. This is a kind of generalization of "tuple notation" to an arbitrary index set J.*

Definitions 1.1.18. *Let J and A_j be sets in Set, with $j \in J$. The **cartesian product** of $\{A_j\}_{j\in J}$ is :*

$$\prod_{j\in J} A_j := \{J\text{-tuples } (x_j)_{j\in J} \text{ of elements of } \cup_{j\in J} A_j \mid \forall j \in J,\, x_j \in A_j\},$$

*and the **coproduct** of $\{A_j\}_{j\in J}$ is:*

$$\coprod_{j\in J} A_j := \bigcup_{j\in J} (A_j \times \{j\})\,.$$

Remark 1.1.19. One can see the coproduct as a disjoint union of non-disjoint sets. With this point of view, we can extend the definition of coproduct to *Top*, using the disjoint union topology (*i.e.* the finest topology that allows the injections $B \hookrightarrow B\coprod C$ and $C \hookrightarrow B\coprod C$ to be continuous). Note that we can also extend the definition of cartesian product the category *Gr* of groups, since a product of groups is still a group where a sum is the sum of each coordonate of the J-tuple. We can say the same with the category *Ab* of abelian groups. Moreover, one can also define the cartesian product in *Top* over topological spaces, endowed with product topology.

Definition 1.1.20. *The **direct sum** of abelian groups $\{A_j\}_{j\in J}$ in Ab is:*

$$\bigoplus_{j\in J} A_j := \left\{J\text{-tuples } (x_j)_{j\in J} \text{ of elements of } \textstyle\prod_{j\in J} A_j \mid\; x_j \neq 0 \text{ for a finite number of } j \text{ in } J\right\} \subset \prod_{j\in J} A_j.$$

Remark 1.1.21. The direct sum $\bigoplus_{j\in J} A_j$ defines an abelian subgroup of $\prod_{j\in J} A_j$, together with the same operation (see the previous remark 1.1.19). We often denote $\sum_{j\in J} x_j$ its elements, where $x_j \in A_j$ for any $j \in J$, because it is a finite sum of non-null elements, which then stays in the group. Moreover, when $J \cong \{1, ..., n\}$ is finite set, we have equality between the direct sum and the cartesian product:

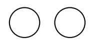
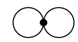
$$\bigoplus_{j\in\{1,\dots,n\}} A_j = \prod_{j\in\{1,\dots,n\}} A_j.$$

Now, we want to define in the category Top_* of pointed spaces something similar to the coproduct we have in Top. That is the reason why we introduce the wedge product.

Definition 1.1.22. *In Top_*, the* ***wedge product*** *$X \vee Y$ of pointed spaces (X, x_0) and (Y, y_0) is:*

$$X \vee Y := X \coprod Y / x_0 \sim y_0.$$

Example. Let us draw $S^1 \coprod S^1$ and $S^1 \vee S^1$ to see the difference:

 $S^1 \coprod S^1$ and  $S^1 \vee S^1$.

Remark 1.1.23. We have the following homeomorphism (with the subspace topology):

$$X \vee Y \;\cong\; (X \times \{y_0\}) \cup (\{x_0\} \times Y) \;=\; \{(x, y) \in X \times Y \mid x = x_0 \text{ ou } y = y_0\}.$$

in $X \times Y$. Moreover, in spite of $X \vee \{*\} \cong X$, we do not have $X \coprod \{*\} \cong X$ since any continuous map $X \to X \coprod \{*\}$ cannot be surjective.

3. Pushout

We want to introduce a powerful tool, pushouts, that will allow us to "glue" some mathematical objects together in a certain sense.

Definition 1.1.24. *Given a category $\mathcal{C}$, and the diagram:*

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & C \\ B, & \end{tikzcd} \tag{1.3}$$

in $\mathcal{C}$, a ***pushout*** *of this diagram is an object $P \in Ob(\mathcal{C})$ together with morphisms $B \to P$ and $C \to P$ such that the following diagram:*

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & C \arrow[d] \\ B \arrow[r] & P, \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

respects a ***universal property****: given any object $D \in Ob(\mathcal{C})$ together with two morphisms $B \to D$ and $C \to D$ in $\mathcal{C}$ such that the diagram:*

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & C \arrow[d] \\ B \arrow[r] & D, \end{tikzcd}$$

commutes, there exists a unique map $\bar{p}$ in $\mathcal{C}$ from P to D such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & C \arrow[d] \arrow[ddr, bend left] & \\ B \arrow[r] \arrow[drr, bend right] & P \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "\exists ! \bar{p}"] & \\ & & D. \end{tikzcd}$$

Notation 1.1.25. *The symbol $\ulcorner$ is there to remind us that P is a pushout.*

Proposition 1.1.26. Given a diagram (1.3) in any category $\mathcal{C}$, a pushout is unique up to isomorphism.

Proof. Let $\mathcal{C}$ be any category. We consider two pushouts P and P' of the diagram (1.3) in $\mathcal{C}$. By definition, there exists a unique morphisms $\bar{p} \in \mathcal{C}(P', P)$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"] & C \arrow[d] \arrow[ddr, bend left] & \\ B \arrow[r] \arrow[drr, bend right] & P' \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "\exists! \bar{p}"] & \\ & & P. \end{tikzcd}$$

Next, changing the roles of P and P', we can see that there is also a unique morphisms $\bar{p}' \in \mathcal{C}(P, P')$ such that the corresponding diagram commutes. We deduce that $\bar{p}'$ is the inverse morphism of $\bar{p}$, which means that $\bar{p}$ is an isomorphism, and thus that $P' \cong P$. $\square$

The following proposition is very important: we compute the pushout in some particular cases. Through it, one can begin to see how powerful are pushouts.

Proposition 1.1.27. Let us consider in a category $\mathcal{C}$ the diagram: We define the equivalence

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & C \\ B, & \end{tikzcd} \tag{1.4}$$

relation $\approx$ as the one generated by $f(a) \approx g(a)$ for $a \in A$. We recall that the pushout is unique up to isomorphism (see 1.1.26).

(i) If $\mathcal{C} := \mathsf{Set}$, then the pushout is $B \coprod_A C := B \coprod C/\approx$.

(ii) If $\mathcal{C} := \mathsf{Top}$, then the pushout is $B \coprod_A C := B \coprod C/\approx$.

(iii) If $\mathcal{C} := \mathsf{Top}_*$, then the pushout is $(B \vee_A C, [b_0]) := (B \vee C/\approx, [b_0])$ where b_0 is the basepoint of B.

(iv) If $\mathcal{C} := \mathsf{Ab}$, then the pushout is $B \oplus_A C := B \oplus C/N$ where N is the smallest normal subgroup of $B \oplus C$ generated by $g(\ker(f))$ and $f(\ker(g))$ (in other words N is generated by the $g(a)$ such that $f(a) = 0$ and by the $f(a)$ such that $g(a) = 0$, where $a \in A$).

In each case, the applications given with the pushouts are the canocial projections denoted $[-]$.

Proof. After showing the first point (i), we will generalize its reasoning to prove the three other points.

(i) Let us show that $B \coprod_A C$ respects the universal property. We take a set $D \in Ob(\mathit{Set})$ and maps $c : C \to D$ and $b : B \to D$ such that the diagram:

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & C \arrow[d, "c"] \\ B \arrow[r, "b"'] & D \end{tikzcd} \tag{1.5}$$

commutes. Let us consider the map $p : B \coprod C \to D$, such that for all $x_B \in B$ we have $p(x_B) = b(x_B)$, and for all $x_C \in C$ we have $p(x_C) = c(x_C)$ (remind from the remark 1.1.19 that we consider the coproduct $B \coprod C$ as the disjoint union of the sets B and C). As the previous diagram (1.5) commutes, and by definition of p, note that $p(f(a)) = b(f(a)) = c(g(a)) = p(g(a))$ for all $a \in A$. As the relation $\approx$ is generated by $f(a) \approx g(a)$ for $a \in A$, we deduce p is $\approx$-invariant and it allows us to define the following map:

$$\bar{p} : \begin{cases} B \coprod_A C & \to & D, \\ [x] & \mapsto & p(x). \end{cases}$$

Now, let us show that the diagram:

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, "f"'] & C \arrow[d, "{[-]}"] \arrow[ddr, bend left, "c"] & \\
B \arrow[r, "{[-]}"'] \arrow[drr, bend right, "b"'] & B \coprod_A C \arrow[dr, dashed, "\bar{p}"] & \\
& & D
\end{tikzcd} \tag{1.6}$$

is commutative. First of all, for all $a \in A$, we obviously have $([.] \circ f)(a) = [f(a)] = [g(a)] = ([.] \circ g)(a)$ by definition of the relation $\approx$. Next, for all $x_B \in B$, we have $(\bar{p} \circ [.])(x_B) = \bar{p}([x_B]) = p(x_B) = b(x_B)$. Finally, for all $x_C \in C$, we have also $(\bar{p} \circ [.])(x_C) = \bar{p}([x_C]) = p(x_C) = c(x_C)$. Thus, the diagram (1.6) is commutative. Now, we need to show that $\bar{p}$ is unique. We consider $\bar{q} : B \coprod_A C \to D$ a map such that the diagram (1.6) above commutes if we replace $\bar{p}$ by $\bar{q}$. Let $[x] \in B \coprod_A C$. If $x \in C$, then we have $\bar{q}([x]) = c(x) = \bar{p}([x])$ by commutativity. If $x \notin C$, then $x \in B$ and we have similarly that $\bar{q}([x]) = b(x) = \bar{p}([x])$. Therefore, we have the equality $\bar{q} = \bar{p}$, which shows the uniqueness of $\bar{p}$. Hence we showed that $B \coprod_A C$ together with the projections $[-]$ satisfies the universal property, and we deduce that it is the pushout of the diagram (1.4).

(II) To show that the pushout in *Top* is also $B \coprod_A C$, it is enough to precise from the previous point (I) that $B \coprod_A C$ is a topological space and that $\bar{p}$ and the projections $[-]$ are continuous. Let us show the first statement. We consider on $B \coprod C$ the disjoint union topology, *i.e.* the finest topology that allows the injections $B \hookrightarrow B \coprod C$ and $C \hookrightarrow B \coprod C$ to be continuous. Then, on the quotient $B \coprod_A C := B \coprod C / \approx$ we consider the quotient topology, *i.e.* the finest topology that allows the canonical projection $B \coprod C \to B \coprod C / \approx$ to be continuous. So $B \coprod_A C$ is a topological space, and by construction of its topology, the projections $[-] : B \to B \coprod_A C$ and $[-] : C \to B \coprod_A C$ are continuous. Now, let us show that $\bar{p} : B \coprod_A C \to D$ is continuous. The map p is continuous because it is defined by the continuous maps b and c on each disjoint composant of the coproduct $B \coprod C$, and it is also $\approx$-invariant as seen in (I). Applying the Universal Property of the Quotient Space, we know there exists a unique continuous map $B \coprod C / \approx \to D$, which is actually $\bar{p}$, such that the following diagram commutes:

$$\begin{tikzcd}
B \coprod C \arrow[r, "p"] \arrow[d, "{[-]}"'] & D \\
B \coprod C / \approx \arrow[ur, dashed, bend right, "\bar{p}."'] &
\end{tikzcd}$$

Thus, the map $\bar{p}$ is continuous, and the pushout is well $B \coprod_A C$ in Top_*.

(III) We want to add some details to the first point (I) to show that $(B \vee_A C, [b_0])$ is the pushout in Top_*. We can see as in (II) that $B \vee_A C$ is a topological space considering the disjoint union and quotient topologies. Moreover, we denote c_0 and d_0 the respective basepoints of C and D, and even though b_0 and c_0 are identified in $B \vee C$, one can note that if now we take p from $B \vee C$ to D, it stays well-defined since the maps c and d preserve the basepoints: $p(b_0) = b(b_0) = d_0 = c(c_0) = p(c_0)$. This implies, as in (I), that $\bar{p} : B \vee_A C \to D$ is also well defined. Now, as in (II), we can observe that $\bar{p}$ and the projections $[-]$ are continuous, but we still need to verify they preserve the basepoints. We also know that $[b_0] = [c_0]$ in $(B \vee_A C, [b_0])$ by definition of $B \vee C$, so the projections $[-] : B \to B \vee_A C$ and $[-] : C \to B \vee_A C$ preserve the basepoints. Concerning $\bar{p}$, it well associates the basepoint $[b_0]$ of $B \vee_A C$ to the basepoint $p(b_0) = b(b_0) = d_0$ of D by commutativity because b is pointed. Hence, in Top_* the pushout is well $(B \vee_A C, [b_0])$.

(IV) Again in this point, we will complete the first point (I) to show that $B \oplus_A C$ is the pushout in *Ab*. We need to prove that $B \oplus_A C$ is an abelian group and that $\bar{p}$ and the projections $[-]$ are group homomorphisms. As explained in the remark 1.1.21, the direct sum $B \oplus C$ is an abelian group. Then, as we quotient by the normal subgroup N, the quotient $B \oplus_A C$ is also an abelian group with the operation defined by $[x] + [y] := [x + y]$ for all $[x], [y] \in B \oplus_A C$. In addition, we can easily see that the projections $[-] : B \to B \oplus_A C$ and $[-] : C \to B \oplus_A C$ are group homomorphisms by definition of the operation in $B \oplus_A C$. Finally, let us take a look at $\bar{p} : B \oplus_A C \to D$. Here we consider a morphism $p : B \oplus C \to D$ slightly different from (I), that associates (x_B, x_C) in $B \oplus C$ to $b(x_B) + c(x_C)$ in D. Recall that N is generated by definition by $g(\ker(f))$ and $f(\ker(g))$, and see that:

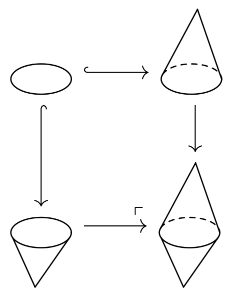
$$\forall a \in \ker(f), \qquad p(g(a)) = b(0) + c(g(a)) \overset{(1.5)}{=} 0 + b(f(a)) = b(0) = 0,$$

and similarly for any a in $\ker(g)$ we have $p(f(a)) = 0$. This means that $N \subseteq \ker(p)$, and we can apply the Universal Property of Quotient Group to get the following commutative diagram:

$$\begin{tikzcd}
B \oplus C \arrow[r, "p"] \arrow[d, "\text{projection}"'] & D \\
B \oplus C/N \arrow[ur, dashed, "\exists ! \bar{p}."']
\end{tikzcd}$$

where $\bar{p}$ is well a group homomorphism. $\square$

Remark 1.1.28. This previous proposition 1.1.27 tells us pushouts can be used to "glue" mathematical objects together. For example, in the following diagram in *Top*, we glue together with $\approx$ the images of the inclusions together:



This pushout is an important example, we will see later more properties about it: this is what we call a suspension (see the remark 1.4.17).

Notation 1.1.29. *When we want to introduce a point without assigning it a name, we will simply write $*$. By the way, observe that the notation Top_* for the category of pointed spaces take on its full meaning.*

Example. In Top_*, considering pointed spaces $(A, x_0) \subset (X, x_0)$, we have the pushout:

$$\begin{tikzcd}
(A, x_0) \arrow[r] \arrow[d, hook] & (\{*\}, *) \arrow[d, hook] \\
(X, x_0) \arrow[r] & (X/A, [x_0]), \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

because we notice that $X \vee_A \{*\} = X \vee \{*\}/\approx \overset{(\star)}{=} X \vee \{*\}/A \cong X/A$, where the equality $(\star)$ is exact since the relation $\approx$ consists in identifying A to a point.

Remark 1.1.30. In any category $\mathcal{C}$, we can find a diagram that admits a pushout. For example, if we take a morphism $f : X \to Y$ of $\mathcal{C}$, we notice we have the following pushout:

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, "f"'] & X \arrow[d, "f"] \\
Y \arrow[r, equal] & Y, \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

where the symbol "=" means that we use the identity *id*. Indeed, let us show that Y respects the universal property. We consider an object D together with two morphisms $c : Y \to D$ and $d : X \to D$ such that the diagram commutes:

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, "f"'] & X \arrow[d, "d"] \\
Y \arrow[r, "c"] & D.
\end{tikzcd} \tag{1.7}$$

We want to find a morphism $\bar{p}$ such that the following diagram commutes:

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, "f"'] & X \arrow[d, "f"] \arrow[ddr, bend left, "d"] & \\
Y \arrow[r, equal] \arrow[drr, bend right, "c"'] & Y \arrow[dr, dashed, "\bar{p}"] & \\
& & D.
\end{tikzcd}$$

We can see with the "lower triangle" that the only possibility for $\bar{p}$ is to be equal to c. Thanks to the commutativity of (1.7), we have $d = c \circ f = \bar{p} \circ f$ which is consistent with the fact that the "upper triangle" must commute. Thus, by construction $\bar{p}$ is unique and the pushout is well Y.

II. Homotopy invariants

In this section, we will define the n-th homotopy group π_n for $n \geq 0$ and see it is a homotopy invariant functor which lands in a group when $n \geq 1$. See the reference [AGP02, pages 59-88] for more details.

1. Homotopy theory

The main problem of homeomorphisms is that they are too rigid, it is usually easier to think up to homotopy. For instance, we will see that we have the homotopy equivalence $D^2 \simeq *$ while they are clearly not homeomorphic: $D^2 \ncong *$. But first, let us introduce a new category in which we will define homotopy.

Notation 1.2.1. *The category Top_{rel} of relative spaces is the category whose objects and morphisms are:*

$$Ob(\mathsf{Top}_{rel}) := \{(X,\, A) \mid \text{the subspace } A \subset X \text{ is furnished with the topology induced by the topological space } X\},$$

$$Mor(\mathsf{Top}_{rel}) := \{f : (X,\, A) \to (Y,\, B) \mid f : X \to Y \text{ is continuous map such that } f(A) \subset B\};$$

and we use in Top_{rel} the usual composition $\circ$ over continuous maps. Such objects $(X,\, A)$ are called ***relative spaces****.*

Remark 1.2.2. Any pointed space $(X,\, x_0)$ in Top_* could be seen as the relative space $(X,\, \{x_0\})$ in Top_{rel}, this is why $\mathsf{Top}_* \subseteq \mathsf{Top}_{rel}$. Similarly, any topological space X in Top could be seen as the relatif space $(X,\, \emptyset)$ in Top_{rel}, tso we also have $\mathsf{Top} \subseteq \mathsf{Top}_{rel}$.

Definitions 1.2.3. *Throughout this project, let be the interval $I := [0,\, 1]$ endowed with the relative topology induced by the usual topology of $\mathbb{R}$.*

(i) *We say any morphisms f and g from $(X,\, A)$ to $(Y,\, B)$ in Top_{rel} are* ***relative homotopic****, and we note $f \simeq_A g$, if there is a continuous map:*

$$H : (X \times I,\, A \times I) \to (Y,\, B),$$

called ***relative homotopy*** *in Top_{rel} (so $H(a,\, t) \in B$ for all $a \in A,\ t \in I$) such that $H(-,\, 0) = f(-)$ and $H(-,\, 1) = g(-)$.*

(ii) *In the particular case of $A = \emptyset = B$, i.e. we work in Top, we sometimes simply denote A and B instead of respectively $(A,\, \emptyset)$ and $(B,\, \emptyset)$, and we say f and g are* ***homotopic*** *if they are relative homotopic. Still in this case, we note $f \simeq g$, and H is simply called* ***homotopy****.*

(iii) *In the case of $A = * = B$, i.e. we work in Top_*, if two continous maps are relative homotopic f and g we say they are* ***pointed homotopic****, we note $f \simeq_* g$ and we call H* ***pointed homotopy****.*

Remark 1.2.4. Each time we want to construct a new relative homotopy H', we need to verify its continuity. Nevertheless, in this project, to stay brief but enough, we will give precisions about the continuity only when it is not clear; indeed, maps will be very often continuous simply by composition of continuous maps. We will adopt the same attitude toward the fact that we need to show that $H'(a,\, t) \in B$ for all $(a,\, t) \in A \times I$ because it will generally come from basic properties.

Proposition 1.2.5. GIVEN TWO RELATIVE SPACES $(X,\, A)$ AND $(Y,\, B)$ IN Top_{rel}, RELATIVE HOMOTOPY $\simeq_A$ IS AN EQUIVALENCE RELATION BETWEEN CONTINUOUS MAPS $(X,\, A) \to (Y,\, B)$.

Proof. First, relative homotopy is reflexive: for any morphism $f : (X,\, A) \to (Y,\, B)$ in Top_{rel}, it is enough to take $H(s,\, t) := f(s)$ for $(s,\, t) \in X \times I$. Furthermore, relative homotopy is symmetric: given morphisms $f,\, g : (X,\, A) \to (Y,\, B)$ in Top_{rel}, if f is relative homotopic to g, and if we note H the corresponding relative homotopy, then the relative homotopy $H'(s,\, t) := H(s,\, 1-t)$ for $(s,\, t) \in X \times I$ give the relation $g \simeq_A f$. Last, relative homotopy is also a transitive relation: if we take three continuous maps $f,\, g,\, h : (X,\, A) \to (Y,\, B)$ in Top_{rel} such that f is relative homotopic to g and g to h with the corresponding relative homotopies H and H', then the relative homotopy H'' defined by $H''(s,\,t) := H(s,\,2t)$ for $(s,\,t) \in X \times [0,\,1/2]$, and by $H''(s,\,t) := H'(s,\,2t-1)$ for $(s,\,t) \in X \times [1/2,\,1]$ gives thes wanted relative homotopy between f and h. $\square$

Notation 1.2.6. *This is why we can take the quotient of $\mathsf{Top}_{rel}((X,A),\,(Y,B))$ by the relative homotopy $\simeq_A$, that we denote:*

$$[(X,\,A),\,(Y,\,B)] := \mathsf{Top}_{rel}((X,\,A),\,(Y,\,B))/\simeq_A,$$

In other words, this is the set of relative homotopy classes of continuous map $(X,\,A) \to (Y,\,B)$ in Top_{rel}, and we denote its elements $[f]$, where $f \in \mathsf{Top}_{rel}((X,\,A),\,(Y,B))$. In particular, if we work on $\mathsf{Top}_* \subseteq \mathsf{Top}_{rel}$, we will prefer the notation $[(X,\,x_0),\,(Y,\,y_0)]_*$ for the quotient, and $[f]_*$ for its elements, where $f \in \mathsf{Top}_*((X,x_0),\,(Y,y_0))$.

Definitions 1.2.7. *We say that two relative spaces (X,A) and (Y,B) in Top_{rel} are **homotopic** and we denote $(X,A) \simeq (Y,B)$ if there are two continuous maps $f : (X,\,A) \to (Y,\,B)$ and $g : (Y,\,B) \to (X,\,A)$ such that:*

$$g \circ f \simeq_A id_{(X,\,A)}, \qquad \text{and} \qquad f \circ g \simeq_B id_{(Y,\,B)}.$$

*We then call f **homotopy equivalence**, and g **homotopy inverse**.*

Definition 1.2.8. *Let $\mathcal{C}$ be any category. A functor $F : \mathsf{Top}_{rel} \to \mathcal{C}$ is a **homotopy invariant** if for all continuous maps $f,\,g : (X,A) \to (Y,\,B)$ in Top_{rel} we have: $\quad f \simeq_A g \;\Rightarrow\; F(f) = F(g)$.*

Proposition 1.2.9. For any category $\mathcal{C}$, if $F : \mathsf{Top}_{rel} \to \mathcal{C}$ is homotopy invariant, and given $(X,\,A) \simeq (Y,\,B)$ in Top_{rel}, then $F(X,\,A)$ and $F(Y,\,B)$ are isomorphic: $\quad F(X,\,A) \cong F(Y,\,B)$.

Proof. As $(X,\,A) \simeq (Y,\,B)$, we can find two continuous maps $f : (X,\,A) \to (Y,\,B)$ and $g : (Y,\,B) \to (X,\,A)$ in Top_{rel} such that $g \circ f \simeq_A id_{(X,\,A)}$ and $f \circ g \simeq_B id_{(Y,\,B)}$. Now, as the functor F is homotopy invariant, we have:

$$F(g) \circ F(f) \;=\; F(g \circ f) \;=\; F(id_X) \;=\; id_{F(X,\,A)},$$

and similarly $F(f) \circ F(g) = id_{F(Y,\,B)}$. Thus, we deduce that $F(f)$ is an isomorphism between $F(X,\,A)$ and $F(Y,\,B)$, which means in other words $F(X,\,A) \cong F(Y,\,B)$. $\square$

Remark 1.2.10. We can similarly define the notion of homotopy invariant contravariant functor, and the the former result still holds.

2. The fundamental group π_1

Let us define the fundamental group π_1 with the notations introduced in 1.2.6.

Definition 1.2.11. *Given a pointed space $(X,\,x_0)$, we define the **fundamental group** π_1 as follows:*

$$\pi_1(X,\,x_0) := [(I,\,\{0,\,1\}),\,(X,\,\{x_0\})] = \mathsf{Top}_{rel}((I,\,\{0,\,1\}),\,(X,\,\{x_0\}))/\simeq_{\{0,\,1\}}\ .$$

In other words, $\pi_1(X,\,x_0)$ is the set of homotopy classes of loops of X based in x_0.

Proposition 1.2.12. Given any relative space $(X,\,A)$ in Top_{rel}, we have the following homotopy invariant functor:

$$[(X,\,A),-] : \begin{cases} \quad\mathsf{Top}_{rel} & \to \quad \mathsf{Set} \\ \quad(Y,\,B) & \mapsto \quad [(X,A),(Y,B)] \\ (f : (Y,\,B) \to (Z,\,C)) & \mapsto \quad (f_* : [(X,A),(Y,B)] \to [(X,A),(Z,C)]), \end{cases}$$

where $f_* : [g] \mapsto [f \circ g]$.

Proof. We split the proof into two steps.

(i) Let us show that $[(X,\,A),-]$ is a functor. Consider in Top_{rel} the continuous maps $f : (Y,\,B) \to (Z,\,C)$ and $g : (Z,\,C) \to (W,\,C)$. On the one hand, we can see that $[(X,\,A),-]$ preserves the composition because for all $[h] \in [(X,A),(Y,B)]$ we have:

$$\begin{aligned} [(X,\,A), f \circ g]([h]) &= (f \circ g)_*([h]) \;=\; [(f \circ g) \circ h] \;=\; [f \circ (g \circ h)] \\ &= f_*([g \circ h]) \;=\; f_*(g_*([h])) \;=\; [(X,\,A), f] \circ [(X,\,A), g]([h]). \end{aligned}$$

On the other hand, we also observe that F preserves the identities since for all $[h] \in [(X,A),(Y,B)]$,

$$[(X,\,A), id_{(Y,\,B)}]([h]) \;=\; (id_{(Y,\,B)})_*([h]) \;=\; [id_{(Y,\,B)} \circ h] \;=\; [h] \;=\; id_{[(X,A),(Y,B)]}[h].$$

(ii) Now let us show that $[(X,\, A), -]$ is homotopy invariant. Consider two morphisms the continuous maps $f, g : (Y,\, B) \to (Z,\, C)$ in $\mathit{Top}_{\mathit{rel}}$ that are homotopic. Denote $H : (Y \times I,\, B \times I) \to (Z,\, C)$ the corresponding relative homotopy, and take $[h] \in [(X, A), (Y, B)]$. We can see that the map $H' : (s,\, t) \mapsto H(h(s),\, t)$ for $(s,\, t) \in X \times I$ is a homotopy between $f \circ h$ and $g \circ h$, which implies $[f \circ h] = [g \circ h]$. Thus we have:

$$\forall [h] \in [(X, A), (Y, B)], \qquad [(X,\, A), f]([h]) \;=\; f_*([h]) \;=\; [f \circ h] \;=\; [g \circ h] \;=\; g_*([h]) \;=\; [(X,\, A), g]([h]),$$

which means that $[(X,\, A), -]$ is homotopy invariant. $\square$

Remark 1.2.13. So, we can deduce that $\pi_1(-) := [(I,\, \{0,\, 1\}),\, -]$ is a homotopy invariant functor.

Remark 1.2.14. One can similarly show that $[-,\, (Y,\, B)]$ is a homotopy invariant contravariant functor $\mathit{Top}_{\mathit{rel}} \to \mathit{Set}$ for any relative space $(Y,\, B)$ in $\mathit{Top}_{\mathit{rel}}$.

Proposition 1.2.15. For any pointed space $(X,\, x_0)$ in Top_*, we have the bijection in Set:

$$\pi_1(X,\, x_0) \cong [(S^1, *),\, (X, x_0)]_*.$$

Proof. Recall from the definition 1.2.11 that $\pi_1(X,\, x_0) := [(I,\, \{0,\, 1\}),\, (X,\, \{x_0\})]$. We claim that there is the following bijection in Set:

$$\mathit{Top}_{\mathit{rel}}((I,\, \{0,\, 1\}),\, (X,\, x_0)) \cong \mathit{Top}_{\mathit{rel}}((I/\{0,\, 1\},\, [0]),\, (X,\, x_0)).$$

Indeed, for any continuous map $f : (I,\, \{0,\, 1\}) \to (X,\, x_0)$ in $\mathit{Top}_{\mathit{rel}}$, considering the equivalence relation $\approx$ on I generated by $0 \approx 1$, we get with the Universal Property of the Quotient Space:

$$\begin{tikzcd}
(I,\, \{0,\, 1\}) \arrow[r, "f"] \arrow[d, "\pi"'] & (X,\, x_0) \\
(I/\approx,\, [0]) \arrow[ur, dashed, bend right, "\exists! \bar{f}"'] &
\end{tikzcd}$$

where $\pi : I \to I/\approx$ is the natural quotient mapping. Conversely, we can associate to any continuous map $\bar{f} : (I/\sim,\, \{0,1\}/\sim) \to (X,\, x_0)$ in $\mathit{Top}_{\mathit{rel}}$ we can associate the map $f := \bar{f} \circ \pi$ in $\mathit{Top}_{\mathit{rel}}((I,\, \{0,\, 1\}),\, (X,\, x_0))$, which makes the map $f \mapsto \bar{f}$ be a bijection by construction. So the claim is verified and, by passage to the quotients, we obtain:

$$\pi_1(X,\, x_0) = [(I,\, \{0, 1\}), (X,\, x_0)] \cong [(I/\{0, 1\},\, [0]), (X,\, x_0)]. \tag{1.8}$$

Moreover, due to the homeomorphism $I/\{0,\, 1\} \cong S^1$, we have the homeomorphism in $\mathit{Top}_{\mathit{rel}}$:

$$(I/\{0, 1\},\, [0]) \cong (S^1,\, *).$$

Hence, we finally deduce that:

$$\pi_1(X,\, x_0) \overset{(1.8)}{\cong} [(I/\{0, 1\},\, [0]), (X,\, x_0)] \overset{1.1.9}{\cong} [(S^1, *),\, (X, x_0)]. \qquad \square$$

3. The n-th homotopy group π_n

In order to generalize the concept of fundamental group π_1, we want to introduce the notion of n-th homotopy group π_n for any non-negative integer n in Top_*. We will also see later a generalization in $\mathit{Top}_{\mathit{rel}}$: the n-th relative homotopy group (see page 34).

Definition 1.2.16. *Let $n \geq 0$. The **n-th homotopy group** π_n of a pointed topological space $(X,\, x_0)$ in Top_* is:*

$$\pi_n(X,\, x_0) := [(S^n,\, *),\, (X,\, x_0)]_*.$$

where $S^0 := \{x \in \mathbb{R} \mid x^2 = 1\} = \{-1,\, 1\}$.

Remark 1.2.17. With the proposition 1.2.15, we can see that this definition gives a generalization of the fundamental group.

Remark 1.2.18. With the proposition 1.2.12, we can notice that π_n is a homotopy invariant functor.

Remark 1.2.19. The computation of $\pi_n(S^m)$ for integers $n, m \geq 0$ is quite complicate, but it fascinates algebraic topologists since it enables to understand better objects that are as basic as the spheres: indeed $\pi_n(S^m)$ describes how spheres of various dimensions can wrap around each other. For instance, we have $\pi_1(S^1) \cong \mathbb{Z}$, and $\pi_n(S^1) \cong 0$ for $n \geq 2$, but also $\pi_2(S^2) \cong \mathbb{Z}$ and $\pi_3(S^2) \cong \mathbb{Z}$. Find more details in the reference [Tod63].

Theorem 1.2.20. Let be $n \geq 1$ a positive integer, and (X, x_0) a pointed topological space. The n-th homotopy group $\pi_n(X, x_0)$ is a group.

Proof. Find the complete proof in appendix. We just give here the main steps.

(i) The quotient set $[(I^n, \partial I^n), (X, x_0)]$ is a group, where ∂I^n is the boundary of the n-dimentional unit cube I^n.

(ii) The spaces $(I^n/\partial I^n, [0])$ and $(S^n, *)$ are in bijection: $(I^n/\partial I^n, [0]) \cong (S^n, *)$.

(iii) The n-th homotopy group $\pi_n(X, x_0)$ is a group. $\square$

Proposition 1.2.21. Up to bijection, the 0-th homotopy group $\pi_0(X, x_0)$ of a pointed space (X, x_0) in Top_* is the set of all path components of X.

Proof. We recall from the definition 1.2.16 that $\pi_0(X, x_0) := [(\{-1, 1\}, 1), (X, x_0)]$. Let $[\gamma] \in \pi_0(X, x_0)$ and $\gamma_1, \gamma_2 \in [\gamma]$. We know that $\gamma_1(1) = x_0 = \gamma_2(1)$, but $\gamma_1(-1)$ and $\gamma_2(-1)$ can take any value in X. As γ_1 and γ_2 are in the same homotopy class, they are homotopic, and we denote H the corresponding relative homotopy. One can notice that $\Gamma(-) := H(-1, -)$ defines a continuous path in X from $\gamma_1(-1)$ to $\gamma_2(-1)$. Thus we can associate the path component $[\Gamma]$ of the constructed path Γ to the class $[\gamma]$ with the map we denote $\theta : [\gamma] \to [\Gamma]$. Reciprocally, let us take any path component $[\Gamma]$. We observe that the maps $\gamma_1 : -1 \mapsto \Gamma(0), 1 \mapsto x_0$ and $\gamma_2 : -1 \mapsto \Gamma(1), 1 \mapsto x_0$ are relative homotopic with the relative homotopy defined by $H(-1, -) := \Gamma(-)$ and $H(1, -) := x_0$. Thus, we can also associate the class $[\gamma_1] = [\gamma_2]$ to $[\Gamma]$, which shows there is an inverse map for θ. That is why θ defines a bijection between $\pi_0(X, x_0)$ and the set of all path components of X. $\square$

Definitions 1.2.22. *We say that a pointed topological space (X, x_0) in Top_* is **contractible** if it is homotopic to a point, i.e. $(X, x_0) \simeq (\{x_0\}, x_0)$.*

Remark 1.2.23. A pointed topological space (X, x_0) is contractible *if, and only if,* we can find morphisms $f : (X, x_0) \to (\{x_0\}, x_0)$ and $g : (\{x_0\}, x_0) \to (X, x_0)$ in Top_* such that $id_X \simeq_{\{x_0\}} g \circ f = g(x_0)$ and $x_0 = id_{\{x_0\}} \simeq_{\{x_0\}} f \circ g = x_0$. And, this condition is equivalent to the following one: the identity id_X is relative homotopic to a constant map.

Proposition 1.2.24. If a pointed space (X, x_0) in Top_* is contractible, then $\pi_n(X, x_0) = 0$ for all $n \geq 0$.

Proof. Let $n \geq 0$ an integer. Thanks to the remark 1.2.18, we know that π_n is a homotopy invariant functor. So, as (X, x_0) is contractible, we can use the proposition 1.2.9 to deduce that $\pi_n(X, x_0) \cong \pi_n(\{x_0\}, x_0)$, which is a trivial set. Thus, $\pi_n(X, x_0) = \{[x \mapsto x_0]\} = 0$. $\square$

4. Weak homotopy equivalences

As we saw in the theorem 1.2.20, for any integer $n \geq 1$, the n-th homotopy group $\pi_n(X, x_0)$ of a pointed space (X, x_0) in Top_* is a group. Now, we will see that weak homotopy equivalences can give us group isomorphisms between those homotopy groups.

Definitions 1.2.25. *We say that a continuous map $f : X \to Y$ in Top is a **weak (homotopy) equivalence** if $X \neq \emptyset$ and for any x_0 in X the induced morphism:*

$$f_* : \begin{cases} \pi_n(X, x_0) & \to & \pi_n(Y, f(x_0)) \\ [h] & \mapsto & f_*([h]) := [f \circ h] \end{cases}$$

is a bijection for $n = 0$, and an group isomorphism for $n \geq 1$. We then denote $X \sim Y$. Moreover, we say that a continuous map $g : (X, A) \to (Y, B)$ in Top_{rel} is a ***(relative) weak (homotopy) equivalence*** *if both $g : X \to Y$ and its restriction $g|_{A \to B}$ are weak homotopy equivalences. We then write $(X, A) \sim (Y, B)$.*

Remark 1.2.26. We saw in the proposition 1.2.21 that $\pi_0(X, x_0)$ is the set of all path components of X. If X and Y are path-connected (*i.e.* we can find a path joining any arbitrary two points), then $\pi_0(X, x_0)$ and $\pi_0(Y, f(x_0))$ are singleton sets, so f_* is trivially a bijection for $n = 0$. More generally, for $n \geq 1$, we would need to verify that f_* is an isomorphism only for a single arbitrary point x_0 in X because if we change the base point, we get a new based space homotopic to the previous one, so we just obtain a homotopy group which is isomomorphic to the previous one since π_n is a homotopy invariant functor. Even more generally, if we do not assume the path-connectedness of X and Y, it is enough to check that f_* is an isomorphism only one element x_0 of each connected component of X.

Remark 1.2.27. The induced morphism f_* is always a group homomorphism for $n \geq 1$. Indeed, instead of working in $\pi_n(X, x_0)$, let us work in one of its isomorphic group: $G := [(I^n, \partial I^n), (X, x_0)]$, together with the operation $\cdot$ seen in the proof of 1.2.20. For $[h]$ and $[k]$ in G, and for all $(s_1, \dots, s_n) \in I^n$, we have:

$$\begin{aligned} f \circ (h+k)(s_1, \dots, s_n) &= f \circ \begin{cases} h(2s_1,\, s_2, \dots, s_n) & \text{for } s_1 \in [0,\, 1/2], \\ k(2s_1 - 1,\, s_2, \dots, s_n) & \text{for } s_1 \in [1/2,\, 1], \end{cases} \\ &= \begin{cases} f \circ h(2s_1,\, s_2, \dots, s_n) & \text{for } s_1 \in [0,\, 1/2], \\ f \circ k(2s_1 - 1,\, s_2, \dots, s_n) & \text{for } s_1 \in [1/2,\, 1], \end{cases} \\ &= ((f \circ h) + (f \circ k))(s_1, \dots, s_n). \end{aligned}$$

Thus, we have that f_* is a group homomorphism:

$$f_*([h] \cdot [k]) \;=\; f_*([h+k]) \;=\; [f \circ (h+k)] \;=\; [(f \circ h) + (f \circ k)] \;=\; [f \circ h] \cdot [f \circ k] \;=\; f_*([h]) \cdot f_*([k]).$$

That is why, when we want to prove that a continuous map f is a weak homotopy equivalence in Top, it is enough to show that f_* is a bijection for all $n \in \mathbb{N}$ and all $x_0 \in X$.

As we can see in the following proposition, the weak homotopy equivalence is a weaker notion than homotopy equivalence on Top_*, which could explain its name.

Proposition 1.2.28. A HOMOTOPY EQUIVALENCE OF POINTED SPACES IS A RELATIVE WEAK HOMOTOPY EQUIVALENCE.

Proof. Let us consider a homotopy equivalence $f : (X, x) \to (Y, y)$ and its homotopy inverse $g : (Y, y) \to (X, x)$ in Top_* corresponding with the relation $(X, x) \simeq (Y, y)$. We have $g \circ f \simeq_{\{x\}} id_{(X,x)}$ and $f \circ g \simeq_{\{y\}} id_{(Y,y)}$. We want to show that the continuous maps $f' : X \to Y$ and $f'|_{\{x\} \to \{y\}}$ associated to $f : (X, x) \to (Y, y)$ are weak homotopy equivalences. We denote $g' : Y \to X$ the continuous map associated to $g : (Y, y) \to (X, x)$. For any x_0 in X and n in $\mathbb{N}$, it is enough to use the fact that π_n is a homotopy invariant functor (see the remark 1.2.18) to see that f'_* is a bijection:

$$g'_* \circ f'_* \;=\; \pi_n(g) \circ \pi_n(f) \;=\; \pi_n(g \circ f) \;=\; \pi_n(id_{(X,x)}) \;=\; id_{\pi_n(X,x)},$$

and, in a same way, $f'_* \circ g'_* = id_{\pi_n(Y,y)}$. So, with the previous remark 1.2.27, the continuous map f' is a weak homotopy equivalence. We can do similarly to show that $f'|_{\{x\}}$ is also a weak homotopy equivalence. Thus f is well a relative weak homotopy equivalence. $\square$

III. Exact sequences of abelian groups

1. Generalities

As long as we stay in the category Ab in this subsection, it is not necessary to repeat that the groups we take are abelian, nor the the morphisms are group homomorphisms. Moreover, the quotient G/H of the group G by the subgroup H is well defined because G is abelian (which implies H is normal in G). Find more details in [Rot09, 2.1].

Definitions 1.3.1. *A* ***(long) exact sequence*** *of abelian groups is a sequence:*

$$\dots \xrightarrow{f_{n+2}} A_{n+1} \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} A_{n-1} \xrightarrow{f_{n-1}} \dots \tag{1.9}$$

in the category Ab such that $\mathrm{Im}(f_{n+1}) = \ker(f_n)$ *for all integer* $n \in \mathbb{Z}$*. A* ***short exact sequence*** *of abelian groups is the following particular case:*

$$0 \longrightarrow A \xrightarrow{\quad f \quad} B \xrightarrow{\quad g \quad} C \longrightarrow 0. \tag{1.10}$$

Remark 1.3.2. A short exact sequence seems to be "finite" and thus seems not to be a long exact sequence, but we can "add" on the right and on the left of the diagram (1.10) as much null abelian group 0 as we want, so that the definition of short exact sequence really matches with the one of long exact sequence. Moreover, one could note that it is not necessary to precise what is the homomorphism $0 \to A$: as far as we want it to map 0 to 0, this morphism is unique. And it is even easier to see that there is only one homomorphism $C \to 0$, namely the constant morphism to 0. A last remark is the following: for such a long exact sequence (1.9), for all integer n, since $\mathrm{Im}(f_{n+1}) = \ker(f_n)$, we necessarily have $f_n \circ f_{n+1} = 0$.

Proposition 1.3.3. The following three statements hold:

(i) A long sequence $0 \to A \overset{f}{\to} B \to \dots$ is exact if, and only if, the morphism f is injective.

(ii) A long sequence $\dots \to B \overset{g}{\to} C \to 0$ is exact if, and only if, the morphism g is surjective.

(iii) A long sequence $0 \to A \overset{h}{\to} C \to 0$ is exact if, and only if, the morphism h is an isomorphism.

Proof.

(i) The sequence $0 \to A \overset{f}{\to} B \to \dots$ is exact *if, and only if,* we have $\ker(f) = \mathrm{Im}(0) = \{0\}$ which is equivalent to f is injective.

(ii) The sequence $\dots \to B \overset{g}{\to} C \to 0$ is exact *if, and only if,* we have $\mathrm{Im}(g) = \ker(0) = C$ which is equivalent to g is surjective.

(iii) The result comes from the pooling of the two previous results. □

Remark 1.3.4. The two first points (i) and (ii) of this proposition tell us that in the short sequence (1.10) is exact *if, and only if,* the morphisms f and g are respectively injective and surjective, and $\mathrm{Im}(f) = \ker(g)$. So it is an equivalent definition. Nonetheless, we do not have the exactness of (1.10) *if, and only if,* the morphism $g \circ f$ is an isomorphism and $\mathrm{Im}(f) = \ker(g)$. Indeed, if $g \circ f$ is an isomorphism and $\mathrm{Im}(f) = \ker(g)$ then the sequence (1.10) is well exact by (iii), but conversely, if the sequence (1.10) is exact, we have with the end of the remark 1.3.2 that $g \circ f$ is constant to 0, which means it cannot be an isomorphism if $C \neq 0$.

Proposition 1.3.5. In Ab, given a long exact sequence:

$$\dots \longrightarrow A_{n+1} \xrightarrow{\quad f_{n+1} \quad} A_n \xrightarrow{\quad f_n \quad} A_{n-1} \longrightarrow \dots$$

and isomorphisms $h_n : A_n \overset{\cong}{\longrightarrow} B_n$ for all integer n, if we have morphisms g_n such that the following diagram commutes:

$$\begin{tikzcd}
\dots \arrow[r] & A_{n+1} \arrow[r, "f_{n+1}"] \arrow[d, "\cong"', "h_{n+1}"] & A_n \arrow[r, "f_n"] \arrow[d, "\cong"', "h_n"] & A_{n-1} \arrow[r] \arrow[d, "\cong"', "h_{n-1}"] & \dots \\
\dots \arrow[r] & B_{n+1} \arrow[r, "g_{n+1}"] & B_n \arrow[r, "g_n"] & B_{n-1} \arrow[r] & \dots
\end{tikzcd} \tag{1.11}$$

then the following long sequence is exact:

$$\dots \longrightarrow B_{n+1} \xrightarrow{\quad g_{n+1} \quad} B_n \xrightarrow{\quad g_n \quad} B_{n-1} \longrightarrow \dots$$

Proof. Let n be an integer. All we need to show it that we have $\mathrm{Im}(g_{n+1}) = \ker(g_n)$. First of all, we can notice that, as the diagram (1.11) commutes, and as h_{n+1} and h_n are isomorphisms, we have:

$$g_{n+1} = h_n \circ f_{n+1} \circ h_{n+1}^{-1}, \qquad \text{and} \qquad g_n = h_{n-1} \circ f_n \circ h_n^{-1}.$$

We consider an $y \in B_n$. We have $y \in \mathrm{Im}(g_{n+1})$ *if, and only if,* we can find an x in B_{n+1} such that $g_{n+1}(x) = y$, *if, and only if,* we have:

$$\begin{aligned} g_n(y) &= g_n \circ g_{n+1}(x) = (h_{n-1} \circ f_n \circ h_n^{-1}) \circ (h_n \circ f_{n+1} \circ h_{n+1}^{-1})(x) \\ &= h_{n-1} \circ (\underbrace{f_n \circ f_{n+1}}_{=0 \text{ due to } 1.3.2})(h_{n+1}^{-1}(x)) = h_{n-1}(0) = 0, \end{aligned}$$

which exactly means that $y \in \ker(g_n)$. Hence the wanted equality. ☐

Remark 1.3.6. This proposition means we can work with exact sequence up to isomorphism.

2. Examples of exact sequences

Proposition 1.3.7. If $H \subseteq G$ is a subgroup of an abelian group G, then we get the following short exact sequence:

$$0 \longrightarrow H \overset{i}{\hookrightarrow} G \xrightarrow{\pi} G/H \longrightarrow 0.$$

where $\pi : G \to G/H$ is the canonical projection.

Proof. As the group G is abelian, we also have that H and G/H are abelian groups. We can see that the inclusion $i : H \hookrightarrow G$ is a group homomorphism since:

$$\forall h_1, h_2 \in H, \qquad i(h_1 + h_2) = h_1 + h_2 = i(h_1) + i(h_2),$$

and it is obviously injective. Furthermore, the canonical projection $\pi : G \to G/H$ is also a group homomorphism:

$$\forall g_1, g_2 \in G, \qquad \pi(g_1 + g_2) = [g_1 + g_2] = [g_1] + [g_2] = \pi(g_1) + \pi(g_2),$$

and it is trivially surjective since each class $[g]$ of G/H contains at least the element g of G. Moreover, we know that for any $[g] \in G/H$, we have $g \in H$ *if, and only if,* we have: $\pi(g) = [g] = 0$, or in other words: $g \in \mathrm{Im}(i) \Leftrightarrow g \in \ker(\pi)$, which means that $\mathrm{Im}(i) = \ker(\pi)$. Hence, with the remark 1.3.4, we obtain that the considered short sequence is exact. ☐

Example. For example, the following short sequence is in particular exact:

$$0 \longrightarrow 3\mathbb{Z} \overset{i}{\hookrightarrow} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/3\mathbb{Z} \longrightarrow 0.$$

Proposition 1.3.8. The two following statements hold:

(i) If the short sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ (1.12) is exact in Ab, then we have the isomorphisms:

$$A \cong \mathrm{Im}(f) \qquad \text{and} \qquad B/\mathrm{Im}(f) \cong C.$$

(ii) Given a subgroup series $T \subseteq S \subseteq M$, there is a short exact sequence:

$$0 \longrightarrow S/T \longrightarrow M/S \longrightarrow M/T \longrightarrow 0. \tag{1.13}$$

Remark 1.3.9. This proposition restates the first and third isomorphism theorems in terms of exact sequences. The first one states that for any group homomorphism $f : G \to G'$ in *Gr*, we have an isomorphism between the image $\mathrm{Im}(f)$ of f and the quotient $G/\ker(f)$: $\mathrm{Im}(f) \cong G/\ker(f)$. The third one insures that for any normal subgroup series $T \triangleleft S \triangleleft M$, we have the isomorphism $(M/S)/(S/T) \cong M/T$.

Proof.

(i) As the sequence (1.12) is exact, we have with the remark 1.3.4 that f and g are respectively injective and surjective and $\mathrm{Im}(f) = \ker(g)$. On the one hand, as $f : A \to B$ is injective, we just need to change the target of f to obtain an isomorphism $A \to \mathrm{Im}(f)$. On the other hand, as $g : B \to C$ is surjective, we have $\mathrm{Im}(g) = C$, so applying the first isomorphism theorem we get:

$$B/\mathrm{Im}(f) = B/\ker(g) \cong \mathrm{Im}(g) = C.$$

(ii) With the third isomorphism theorem, we know that $(M/S)/(S/T)$ is isomorphic to M/T, and then we obtain the exact sequence (1.13) with the proposition 1.3.7 applied with $H := S/T$ and $G := M/S$. $\square$

Proposition 1.3.10. Given two groups A and C in Ab, the natural inclusion $i : A \hookrightarrow A \oplus C$, $a \mapsto (a, 0)$ and projection $\pi : A \oplus C \to C$, $(a, c) \mapsto c$ define a short exact sequence:

$$0 \longrightarrow A \xhookrightarrow{i} A \oplus C \xrightarrow{\pi} C \longrightarrow 0.$$

Remark 1.3.11. Even if i is not exactly an inclusion, we will use its symbol $\hookrightarrow$ and we will call it inclusion because it is an inclusion up to isomorphism. In addition, we have $A \oplus C = A \times C$ (see 1.1.21), and it an abelian group with the usual operation $+$ defined by:

$$\forall (a, c), (a', c') \in A \oplus C, \qquad (a, c) + (a', c') := (a +_A a', c +_C c'),$$

where $+_A$ and $+_C$ are the operation over A and C respectively.

Proof. We can see that the inclusion i is a group homomorphism because:

$$\forall a, a' \in H, \qquad i(a + a') = (a + a', 0) = (a, 0) + (a', 0) = i(a) + i(a'),$$

and it is clearly injective. Moreover, the projection π is also a group homomorphism:

$$\forall (a, c), (a', c') \in A \oplus C, \qquad \pi((a, c) + (a', c')) = \pi(a +_A a', c +_C c') = c + c' = \pi(a, c) + \pi(a', c'),$$

and it is surjective since $\pi(0, c) = c$ for any $c \in C$. Finally, we have:

$$x \in \mathrm{Im}(i) \quad \Leftrightarrow \quad \exists a \in A,\ x = i(a) = (0, a) \quad \Leftrightarrow \quad \pi(x) = 0 \quad \Leftrightarrow \quad x \in \ker(\pi),$$

which means that $\mathrm{Im}(i) = \ker(\pi)$. Thus, we conclude using the remark 1.3.4 that the considered short sequence is exact. $\square$

3. Split exact sequence

Proposition 1.3.12. Splitting Lemma. Given a short exact sequence in Ab:

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0, \tag{1.14}$$

the following three statements are equivalents:

(i) **Left split**: there exists a homomorphism $r : B \to A$, that we call **retraction** of f, such that:

$$r \circ f = id_A,$$

(ii) **Right split**: there exists a homomorphism $s : C \to B$, that we call **section** of g, such that:

$$g \circ s = id_C,$$

(iii) **Direct sum**: there is an isomorphism $h : B \to A \oplus C$ such that the following diagram commutes:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r, "g"] \arrow[d, "\cong"', "h"] & C \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & A \arrow[r, hook, "i"] & A \oplus C \arrow[r, "\pi"] & C \arrow[r] & 0 \end{tikzcd} \tag{1.15}$$

keeping the same notations as before for the natural inclusion i and projection π.

Definition 1.3.13. *We call* ***split exact sequence*** *a short exact sequence* (1.14) *such that one of the 3 previous conditions is verified.*

Proof. We will show the equivalences (i) $\Leftrightarrow$ (iii) and (ii) $\Leftrightarrow$ (iii).

(III) $\Rightarrow$ (I) Assume (III), we have $h \circ f = i$. So, if we consider the natural projection $\pi' : A \oplus C \to A$, we notice that the group homomorphism $r := \pi' \circ h : B \to A$ is a retraction of f, because:

$$\forall a \in A, \qquad r \circ f(a) \;=\; \pi' \circ h \circ f(a) \;=\; \pi' \circ i(a) \;=\; \pi'(a,\, 0) \;=\; a \;=\; id_A(a).$$

(III) $\Rightarrow$ (II) Reasoning as in the previous point, we have the section $s := h^{-1} \circ i'$ of g, where $i' : C \hookrightarrow A \oplus C$.

(I) $\Rightarrow$ (III) We assume there is a retraction $r : B \to A$ of f. We consider the following morphism:

$$h : \begin{cases} B & \to & A \oplus C \\ b & \mapsto & (r(b),\, g(b)) \end{cases}$$

which is a group homomorphism since r and g are so. Observe h is injective, because, for any $b \in B$,

$$h(b) = (0,\,0) \quad \Leftrightarrow \quad \begin{cases} r(b) &= 0 \\ g(b) &= 0 \end{cases} \quad \Leftrightarrow \quad \begin{cases} r(b) = 0 \\ b \in \ker(g) = \mathrm{Im}(f) \end{cases} \quad \Leftrightarrow \quad \begin{cases} r(b) = 0 \\ \exists a \in A,\ b = f(a) \end{cases}$$
$$\Leftrightarrow \quad \begin{cases} r(f(a)) = 0 = a \\ \exists a \in A,\ b = f(a) = f(0) = 0 \end{cases} \quad \Leftrightarrow \quad b = 0.$$

Now, let us show that the morphism h is also surjectif. Let $(a,\,c) \in A \oplus C$. Thanks to the exactness of the short sequence (1.14), the morphism g is surjective, which means we can find an $b \in B$ such that $g(b) = c$. And even much more: notice that for any $x \in A$, we have $g(b + f(x)) = g(b) + g(f(x)) = c$ because $g \circ f = 0$ (see the end of the remark 1.3.4). We would like to find a particular $x_0 \in A$, such that we have in addition $r(b + f(x_0)) = a$. In other words, we want:

$$a = r(b + f(x_0)) = r(b) + r(f(x_0)) = r(b) + x_0 \qquad \Leftrightarrow \qquad x_0 = a - r(b).$$

That is why we take such an $x = x_0$, and we obtain:

$$h(b + f(x)) \;=\; (r(b + f(x)),\, g(b + f(x))) \;=\; (a,\, c),$$

which means h is well surjective. Furthermore, the diagram (1.15) commutes because:

$$\forall a \in A, \qquad h \circ f(a) \;=\; (r(f(a)),\, g(f(a))) \;=\; (a,\, 0) \;=\; i(a),$$

and:

$$\forall b \in B, \qquad \pi \circ h(b) \;=\; \pi(r(b),\, g(b)) \;=\; g(b),$$

Therefore, we showed (I) $\Rightarrow$ (III).

(II) $\Rightarrow$ (III) Suppose there is a section $s : C \to B$ of g. Let us consider the following morphism:

$$h' : \begin{cases} A \oplus C & \to & B \\ (a,\, c) & \mapsto & f(a) + s(c) \end{cases}$$

which is a group homomorphism as f and s are so. We want to show h' is an isomorphism. Let us take $(a,\, c) \in A \oplus C$ such that $0 = h'(a,\, c) := f(a) + s(c)$. We apply g to this equation, and we get:

$$0 \;=\; g(0) \;=\; g(f(a)) + g(s(c)) \;=\; 0 + c \;=\; c.$$

In particular, we then have $f(a) = f(a) + s(c) = 0$. Due to the exactness of the short sequence (1.14), the morphism f is injective, so $a = 0$. To sum up, we have $(a,\, c) = (0,\, 0)$ and we showed h' is injective. Now, let us show that h' is surjective. Let $b \in B$. We notice that b can be decompose into the sum of $b - s(g(b))$ and $s(g(b))$, where the first element is in the kernel $\ker(g)$ of g:

$$g(b - s(g(b))) = g(b) - (g \circ s)(g(b)) = g(b) - g(b) = 0.$$

So, as $\ker(g) = \mathrm{Im}(f)$, we can find an $a \in A$ which verifies $f(a) = b - s(g(b))$. Thus, if we denote $c := g(b)$, we get $h'(a,\, c) = b$, which means h' is well surjective. Hence, h' is an isomorphism, and we obtain the wanted isomorphism $h : B \to A \oplus C$ being the inverse isomorphism of h', knowing moreover that the diagram (1.15) commutes:

$$\forall a \in A, \quad h'(i(a)) = h'(a,\, 0) = f(a) \qquad \Leftrightarrow \qquad h^{-1} \circ i = h' \circ i = f \qquad \Leftrightarrow \qquad i = h \circ f,$$

and:

$$\forall (a,\, c) \in A \oplus C, \quad g \circ h'(a,\, c) = g \circ f(a) + g \circ s(c) = c = \pi(a,\, c) \quad \Leftrightarrow \quad g \circ h^{-1} = g \circ h' = \pi,$$
$$\Leftrightarrow \quad g = \pi \circ h. \qquad \square$$

IV. Eilenberg-Steenrod axioms of homology theory

The central key of homology theory is its axioms, also known as **Eilenberg-Steenrod axioms**. They were introduced in [ES45] by Samuel Eilenberg and Norman E. Steenrod in 1945, but the language used was different from the one of categories we use here.

1. Axioms

We saw previously that in the category $\mathsf{Top}_{\mathsf{rel}}$ there is a way to "pass" from a function to another one: the relative homotopy. Now, we would like to define a similar notion that would allow us to "pass" from a functor to another one: the natural transformation.

Definition 1.4.1. *Consider two functors $F, G : \mathcal{C} \to \mathcal{D}$ between categories $\mathcal{C}$ and $\mathcal{D}$. A* ***natural transformation*** *$\lambda : F \Rightarrow G$ from F to G is a class $(\lambda_X : F(X) \to G(X))_{X \in Ob(\mathcal{C})}$ of morphisms of $\mathcal{D}$ indexed by the objects of $\mathcal{C}$ such that, for any morphism $f : X \to Y$ in $\mathcal{C}$, we have $\lambda_Y \circ F(f) = G(f) \circ \lambda_X$, or, in other terms, the following diagram commutes:*

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ {\scriptstyle \lambda_X}\downarrow & & \downarrow{\scriptstyle \lambda_Y} \\ G(X) & \xrightarrow[G(f)]{} & G(Y). \end{array}$$

Definition 1.4.2. *An* ***excisive triad*** *$(X;\, A,\, B)$ is a triad where A and B are subspaces of a topological space X, endowed with the respective induced topologies, such that $\mathring{A} \cup \mathring{B} = X$.*

Remark 1.4.3. Contrary to the triple $(X,\, A,\, B)$ of topological spaces $B \subseteq A \subseteq X$ that we will use in the next subsection, an excisive triad does not require $B \subseteq A$.

Definition 1.4.4. *Let $U : \mathsf{Top}_{\mathsf{rel}} \to \mathsf{Top}_{\mathsf{rel}}$ be a functor which sends any pair $(X,\, A)$ to the pair $(A,\, \emptyset)$ in $\mathsf{Top}_{\mathsf{rel}}$, and any continuous map $f : (X,\, A) \to (Y,\, B)$ to its restriction $(A,\, \emptyset) \to (B,\, \emptyset)$. A* ***(generalized) homology theory*** *E_* is a family $\{E_n : \mathsf{Top}_{\mathsf{rel}} \to \mathsf{Ab}\}_{n \in \mathbb{Z}}$ of homotopy invariant functors E_n, together with a family $\{\partial_n : E_n \Rightarrow E_{n-1} \circ U\}_{n \in \mathbb{Z}}$ of natural transformations ∂_n, such that the following four axioms are satisfied:*

(H1) ***Exactness:*** *for any pair $(X,\, A)$ in $\mathsf{Top}_{\mathsf{rel}}$, let $i : (A,\, \emptyset) \hookrightarrow (X,\, \emptyset)$ and $j : (X,\, \emptyset) \hookrightarrow (X,\, A)$ be inclusions. The following long sequence is exact in the category Ab of the abelian groups:*

$$\ldots \longrightarrow E_n(A,\, \emptyset) \xrightarrow{E_n(i)} E_n(X,\, \emptyset) \xrightarrow{E_n(j)} E_n(X,\, A) \xrightarrow{\partial_{n\,(X,\, A)}} E_{n-1}(A,\, \emptyset) \longrightarrow \ldots$$

(H2) ***Excision:*** *for any excisive triad $(X;\, A,\, B)$, the inclusion $i : (A,\, A \cap B) \hookrightarrow (X,\, B)$ induces isomorphisms in Ab for all integer n:*

$$E_n(i) : E_n(A,\, A \cap B) \xrightarrow{\cong} E_n(X,\, B).$$

(H3) ***Additivity:*** *for any collection $\{(X_j,\, A_j)\}_{j \in J}$ of pairs in $\mathsf{Top}_{\mathsf{rel}}$, the inclusions $i_{j_0} : (X_{j_0},\, A_{j_0}) \hookrightarrow \coprod_{j \in J}(X_j,\, A_j)$ for $j_0 \in J$ induce isomorphisms in Ab for all integer n:*

$$\sum_{j \in J} E_n(i_j) : \left\{ \begin{array}{ccc} \bigoplus_{j \in J} E_n(X_j,\, A_j) & \xrightarrow{\cong} & E_n\left(\coprod_{j \in J}(X_j,\, A_j)\right), \\ \sum_{j \in J} x_j & \longmapsto & \sum_{j \in J} E_n(i_j)(x_j). \end{array} \right.$$

(H4) ***Invariance with weak equivalences:*** *If $f : (X,\, A) \to (Y,\, B)$ is a relative weak homotopy equivalence in $\mathsf{Top}_{\mathsf{rel}}$, then we have the following isomorphisms in Ab for any integer n:*

$$E_n(f) : E_n(X,\, A) \xrightarrow{\cong} E_n(Y,\, B).$$

We also say that the homology theory E_* is ***ordinary*** *if moreover the following axiom is verified:*

(H5) ***Dimension:*** *there exists an abelian group A such that:*

$$E_n(\{*\},\, \emptyset) = \left\{ \begin{array}{ll} A & \text{if } n = 0, \\ 0 & \text{otherwise}, \end{array} \right.$$

where $*$ designates a point.

Remarks 1.4.5. The natural transformations ∂_n (for $n \in \mathbb{Z}$) are well defined because, as we saw in the remark 1.1.6, the composition of functors is also a functor, so $E_{n-1} \circ U$ is a functor. Moreover, the axioms (H2) and (H3) are also well defined because respectively $(A,\, A \cap B)$ and $\coprod_{j \in J}(X_j,\, A_j) = \left(\coprod_{j \in J} X_j,\, \coprod_{j \in J} A_j\right)$ are relative topologies in Top_{rel}. Let us introduce a couple of notations. When the integer n considered is understood, we will only write $E_*(-)$ instead of $E_n(-)$ and for any continuous map f in the category Top_{rel}, we will often prefer replacing $E_n(f)$ simply by f_*. Moreover, if the relative topology $(X,\, A)$ is also clear, we will prefer ∂_n or even ∂ instead of $\partial_{n(X,\, A)}$. We also write $E_n(X)$ for $E_n(X,\, \emptyset)$ where X is a topological space in Top. Furthermore, if an homology theory E_* is ordinary for a certain abelian group A, then we use to denote $H_*(-;\, A) := E_*(-)$, or even $H_*(-)$ when we want to lighten notations.

Remark 1.4.6. Even though these axioms do not guarantee *a priori* that we can find an ordinary homology theory $H_*(-;\, A)$ for any abelian group A, we will see at the end of the project that the Dold-Thom theorem assures the existence of ordinary homology theory $H_*(-;\, \mathbb{Z})$ in the particular case of $\mathbb{Z}$. More precisely, we use in the Dold-Thom theorem a reduced ordinary homology theory $\tilde{H}_*$, but we will see in the theorem 3.1.8 that there is a one to one relation between a reduced and a non-reduced ordinary homology theories $\tilde{H}_*$ and H_*.

Remark 1.4.7. As explained in [ES52, page 13], we can assume *w.l.o.g.* that $\tilde{H}_n(-) = 0$ for $n < 0$. Moreover, one can notice thanks to the additivity axiom (H3) that $E_*(\emptyset) \cong 0$ because:

$$E_*(\emptyset) \oplus E_*(\emptyset) \overset{(\text{H3})}{\cong} E_*\left((\emptyset,\, \emptyset) \coprod (\emptyset,\, \emptyset)\right) = E_*(\emptyset) \cong E_*(\emptyset) \oplus \{0\}.$$

2. Exact sequences of homology, suspension and reduced cone

We would like to introduce a new tool that will help us to compute $E_*(X)$ in terms of $E_*(A)$, $E_*(B)$ and $E_*(A \cap B)$ for any excisive triad $(X;\, A,\, B)$: the Mayes-Vietoris sequence. But before, let us get another interesting exact sequence. Find it in [May99, page 112].

Proposition 1.4.8. Let E_* be any homology theory with the same notations as in the definition 1.4.4, and let $(X,\, A,\, B)$ be a triple of topological spaces $B \subseteq A \subseteq X$, where A and B are respectively endowed with the topologies induced by X on A and B (so both $(X,\, A)$, (X, B) and $(A,\, B)$ are relative topologies). Consider the inclusions $i : (A,\, B) \hookrightarrow (X,\, B)$, $j : (X,\, B) \hookrightarrow (X,\, A)$ and $k : (A,\, \emptyset) \hookrightarrow (A,\, B)$; and for all integer n, define the group homomorphism λ_n as the composite:

$$\lambda_n : E_n(X,\, A) \xrightarrow{\partial_n} E_{n-1}(A) \xrightarrow{E_{n-1}(k)} E_{n-1}(A,\, B),$$

Then, the following sequence is exact in Ab:

$$\ldots \longrightarrow E_n(A,\, B) \xrightarrow{i_*} E_n(X,\, B) \xrightarrow{j_*} E_n(X,\, A) \xrightarrow{\lambda_n} E_{n-1}(A,\, B) \longrightarrow \ldots \qquad \square$$

Find the following theorem in [May99, page 113].

Theorem 1.4.9. Let E_* be any homology theory with the same notations as in the definition 1.4.4. We consider an excisive triad $(X;\, A,\, B)$ and the intersection $C := A \cap B$. We denote the following inclusions: $i : (C,\, \emptyset) \hookrightarrow (A,\, \emptyset)$, $j : (C,\, \emptyset) \hookrightarrow (B,\, \emptyset)$, $k : (A,\, \emptyset) \hookrightarrow (X,\, \emptyset)$, $l : (B,\, \emptyset) \hookrightarrow (X,\, \emptyset)$ and $m : (X,\, \emptyset) \hookrightarrow (X,\, B)$. Let us also consider for any integer n the homomorphisms:

$$\psi_n : \left\{ \begin{array}{ccc} E_n(C) & \to & E_n(A) \oplus E_n(B), \\ c & \mapsto & (i_*(c),\, j_*(c)), \end{array} \right. \qquad \phi_n : \left\{ \begin{array}{ccc} E_n(A) \oplus E_n(B) & \to & E_n(X), \\ (a,\, b) & \mapsto & k_*(a) - l_*(b), \end{array} \right.$$

and $\Delta_n : E_n(X) \to E_{n-1}(C)$ the composite:

$$E_n(X) \xrightarrow{m_*} E_n(X,\, B) \overset{\text{(H3)}}{\cong} E_n(A, C) \xrightarrow{\partial_n} E_{n-1}(C)$$

Then, the following sequence is exact and is called **Mayes-Vietoris sequence** associated to the excisive tirad $(X;\, A,\, B)$ in Ab:

$$\ldots \longrightarrow E_n(C) \xrightarrow{\psi_n} E_n(A) \oplus E_n(B) \xrightarrow{\phi_n} E_n(X) \xrightarrow{\Delta_n} E_{n-1}(C) \longrightarrow \ldots \qquad \square$$

To reveal the strength of the Mayer-Vietoris sequence for an ordinary homology theory $H_*(-;\, A)$, we will use it to compute $H_n(S^m;\, A)$ for any integers $n,\, m \geq 0$. But, just before, let us define two fundamental constructions: suspension and reduced cone. Recall, we denote $I := [0,\, 1]$ the unit interval endowed with the induced subspace topology by the usual one on $\mathbb{R}$. We fix 0 to be its based point.

Definition 1.4.10. *In Top_* the* ***(reduced) suspension*** ΣX *of a based space $(X,\, x_0)$ is the following pushout:*

$$\begin{tikzcd} (X \times \{0,\, 1\}) \cup (\{x_0\} \times I) \arrow[r, "\exists! f"] \arrow[d, hook, "i"'] & \{*\} \arrow[d, "{[-]}"] \\ X \times I \arrow[r, "{[-]}"'] \arrow[ur, phantom, "\ulcorner"] & \Sigma X \end{tikzcd} \tag{1.16}$$

and we denote $x \wedge t := h(x,\, t) \in \Sigma X$ for any $(x,\, t) \in X \times I$.

Remark 1.4.11. Up to isomorphism, as the pushout is unique (see 1.1.26), we have a unique suspension ΣX associated to a pointed space $(X,\, x_0)$. Note that, as long as basepoints are clear, we will not precise them to lighten the notations. For example, in the $X \times I$ the basepoint is $(x_0,\, 0)$, *i.e.* the couple of the basepoints of X and of I.

Remark 1.4.12. We denote $A := (X \times \{0,\, 1\}) \cup (\{x_0\} \times I)$. In the pushout (1.16) just above, we have $\Sigma X = (X \times I) \vee_A \{*\}$ in Top_* due to the proposition 1.1.27. It means we "glue" together the images of i and f, or in other words, we associate in $X \times I$ all point of A to one point $*$. So, the suspension ΣX is no more than:

$$\Sigma X = (X \times I)/_A = (X \times I)/_{(X \times \{0,\, 1\}) \cup (\{x_0\} \times I)}.$$

Moreover, as h is pointed, the basepoint of ΣX is $h(x_0,\, 0) =: x_0 \wedge 0$ because $(x_0,\, 0)$ is the basepoint of $X \times I$. But as $(x_0,\, 0) \in A$ and as we quotient $X \times I$ by A to get ΣX, the basepoint of ΣX could be in fact written as any $x \wedge t$ such that $(x,\, t) \in A$.

Definition 1.4.13. *In Top_* the* ***reduced cone*** CX *of a based space $(X,\, x_0)$ is the following pushout:*

$$\begin{tikzcd} (X \times \{1\}) \cup (\{x_0\} \times I) \arrow[r, "\exists!"] \arrow[d, hook] & \{*\} \arrow[d, "{[-]}"] \\ X \times I \arrow[r, "{[-]}"'] \arrow[ur, phantom, "\ulcorner"] & CX \end{tikzcd} \tag{1.17}$$

Remark 1.4.14. Similarly as in the remark 1.4.12, if we denote $B := (X \times \{1\}) \cup (\{x_0\} \times I)$ we have with the reduced cone:

$$CX = (X \times I)/_B = (X \times I)/_{(X \times \{1\}) \cup (\{x_0\} \times I)}, \tag{1.18}$$

and its basepoint is $[(x_0, 0)]$, but also any $[(x, t)]$ such that $(x, t) \in B$.

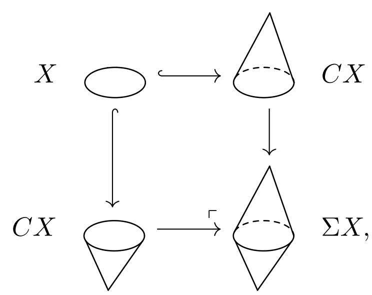
Remark 1.4.15. When X is the circle S^1, one can see that the reduced cone CX is an hemisphere, which is homeomorphic to a cone. That is why we gave to CX such a name. Another explanation would be that the (non-reduced) cone (which is defined as $X \times I/X \times \{1\}$) is exactly a cone.

Remark 1.4.16. In a sense, one can say we have the inclusion $X \hookrightarrow CX$ because the continuous map:

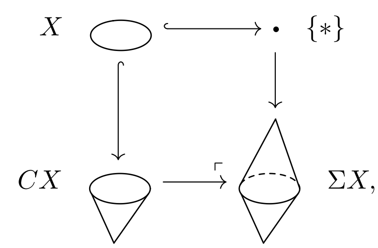
$$\alpha \left\{ \begin{array}{ccccc} X & \cong & X \times \{0\} & \to & CX, \\ x & \mapsto & (x, 0) & \mapsto & [(x, 0)], \end{array} \right.$$

is an embedding, *i.e.* an homeomorphism into its image. Indeed α is clearly injective since, with the quotient (1.18), the only point of $X \times \{0\}$ affected by the identification in CX is $(x_0, 0)$, the other point are not affected by the identification.

Remark 1.4.17. Note that we can express the suspension ΣX using reduced cones CX as follows:



and



where the inclusion $X \hookrightarrow CX$ comes from the remark 1.4.16. The first pushout can be obtained just seeing pushout as a "glue" tool, and the second one results from the proposition 1.1.27 applied to the isomorphism:

$$\begin{array}{rcl} \Sigma X & \cong & X \times I/_{(X \times \{0, 1\}) \cup (\{x_0\} \times I)} \\ & \cong & \left(X \times I/_{(X \times \{1\}) \cup (\{x_0\} \times I)}\right)/_{X \times \{0\}} \\ & \cong & CX/X \\ & \cong & (CX \vee \{*\})/X \\ & \cong & CX \vee_X \{*\}. \end{array}$$

Note moreover that the latter pushout implies the following isomorphism:

$$\Sigma X \cong CX/X.$$

Proposition 1.4.18. For any pointed space (X, x_0) in Top_*, the reduced cone CX is contractible, and the suspension ΣX is path-connected.

Proof. To show that CX is contractible, we need to find a relative homotopy $H : CX \times I \to CX$ from id_{CX} to a constant map (see the remark 1.2.23). We define the following map:

$$h : \left\{ \begin{array}{ccc} (X \times I) \times I & \to & CX \\ ((x, t), s) & \mapsto & [(x, (1-s)t + s)] \end{array} \right.$$

which is continuous by composition of continuous maps. We will apply the Universal Property of Quotient Space in order to have the following commutative diagram:

$$\begin{tikzcd} (X \times I) \times I \arrow[r, "h"] \arrow[d, "{[-] \times id}"'] & CX \\ CX \times I \arrow[ur, dashed, "\exists! H"'] & \end{tikzcd} \quad \text{where} \quad H : \left\{ \begin{array}{ccc} CX \times I & \to & CX, \\ ([(x, t)], s) & \mapsto & h((x, t), s). \end{array} \right.$$

The map $[-] \times id$ is the natural projection from $(X \times I) \times I$ to $CX \times I := ((X \times I) \times I)/\approx$, where the equivalence relation $\approx$ identifies the elements of $(X \times \{1\}) \cup (\{x_0\} \times I)$ on the composant $X \times I$, and it is the equality on the composant I. Let us verify f is $\approx$-invariant. We consider two different points $((x, t), s) \approx ((x', t'), s')$ in $(X \times I) \times I$. We have $s = s'$, and (x, t) and (x', t') belong to $(X \times \{1\}) \cup (\{x_0\} \times I)$. This induces:

$$\begin{aligned} h((x, t), s) &= \begin{cases} [(x, 1)] & \text{if } (x, t) \in X \times \{1\} \\ [(x_0, (1-s)t + s)] & \text{if } (x, t) \in \{x_0\} \times I \end{cases} \\ &= [(x_0, 0)] \\ &= \begin{cases} [(x', 1)] & \text{if } (x', t') \in X \times \{1\} \\ [(x_0, (1-s')t' + s')] & \text{if } (x', t') \in \{x_0\} \times I \end{cases} \\ &= h((x', t'), s') \end{aligned}$$

Thus h is $\approx$-invariant and we can apply the Universal Property of Quotient Space: in particular, the map $H : CX \times I \to CX$ is in Top_*. Hence, this continuous map H is the wanted relative homotopy since for all $[(x, t)] \in CX$,

$$H(-, 0)([(x, t)]) = h((x, t), 0) = [(x, t)] = id_{CX}([(x, t)]),$$
$$H(-, 1)([(x, t)]) = h((x, t), 1) = [(x, 1)] = [(x_0, 1)],$$

To this end, the reduced cone CX is contractible.
Next, to show that ΣX is path-connected, let us show that we can join its basepoint $x_0 \wedge 0$ to any of its points $x \wedge t \in \Sigma X$. The path we consider is the following one:

$$\gamma_{x \wedge t} : \begin{cases} I & \to & \Sigma X, \\ s & \mapsto & x \wedge \big((1-s)t\big), \end{cases}$$

which is in Top_* by composition of pointed continuous maps. We recognize it is the wanted path because in $s = 0$ we get $x \wedge t$ and in $s = 1$ we get $x_0 \wedge 0$. Consequently, the suspension ΣX is path-connected. □

The following result is very important to compute $H_m(S^n)$ for any integers $m, n \geq 0$. Note that, with the following proposition, we could define inductively all the multidimentional spheres from S^0 thanks to the suspension Σ.

Proposition 1.4.19. For any integer $n \geq 0$,

$$\Sigma S^n \cong S^{n+1}.$$

Proof. Let $n \in \mathbb{N}$. We regard S^n and S^{n+1} as sitting in $\mathbb{R}^{n+2}$ together with the usual norm $||.||$:

$$S^{n+1} := \{x \in \mathbb{R}^{n+2} \mid ||x|| = 1\} \qquad \text{and} \qquad S^n := \{x \in \mathbb{R}^{n+2} \mid ||x|| = 1 \text{ and } x_{n+2} = 0\},$$

where S^n is the equator of the unit sphere S^{n+1}. We also consider the disk D^{n+1} in the equatorial plane, and the upper and lower hemisphere H_+^{n+1} and H_-^{n+1} as follows:

$$D^{n+1} := \{x \in \mathbb{R}^{n+2} \mid ||x|| \leq 1 \text{ and } x_{n+2} = 0\},$$
$$H_+^{n+1} := \{x \in \mathbb{R}^{n+2} \mid ||x|| = 1 \text{ and } x_{n+2} \geq 0\},$$
$$H_-^{n+1} := \{x \in \mathbb{R}^{n+2} \mid ||x|| = 1 \text{ and } x_{n+2} \leq 0\}.$$

For all the set defined just above, the basepoint is $s_0 := (1, 0, \ldots, 0) \in \mathbb{R}^{n+2}$ and the topology is the subspace one. One can say there are homeomorphisms in Top_* defined as follows:

$$p_+ : \begin{cases} D^{n+1} & \to & H_+^{n+1} \\ (x_1, \ldots, x_{n+1}, 0) & \mapsto & \big(x_1, \ldots, x_{n+1}, \sqrt{1-v}\big) \end{cases}$$
$$p_- : \begin{cases} D^{n+1} & \to & H_+^{n+1} \\ (x_1, \ldots, x_{n+1}, 0) & \mapsto & \big(x_1, \ldots, x_{n+1}, -\sqrt{1-v}\big) \end{cases}$$

where $v := \sum_{i=1}^{n+1} x_i$, on account of they are continuous and bijective between compact Hausdorff objects. Now, as the disk D^{n+1} is convex, for any $x \in S^n \subset D^{n+1}$ and $t \in I$, we have that $tx + (1-t)s_0$ is in D^{n+1}. So the following map:

$$h : \begin{cases} S^n \times I & \to & S^{n+1} \\ (x, t) & \mapsto & \begin{cases} p_-(2tx + (1-2t)s_0) & \text{if } 0 \leq t \leq 1/2 \\ p_+(2(1-t)x + (2t-1)s_0) & \text{if } 1/2 \leq t \leq 1 \end{cases} \end{cases}$$

is well defined (also when $t = 1/2$ because we would have $p_-(x) = p_+(x)$ for $x \in S^n$), and continuous by composition of continuous maps. It is also invariant over $(S^n \times \{0, 1\}) \cup (\{s_0\} \times I)$ because, for any $(x, t) \in S^n \times I$:

$$\begin{array}{rcccl} h(x, 0) & = & p_-(s_0) & = & s_0, \\ h(x, 1) & = & p_+(s_0) & = & s_0, \end{array} \qquad h(s_0, t) \;=\; \left\{ \begin{array}{ll} p_-(s_0) & \text{if } 0 \leq t \leq 1/2 \\ p_+(s_0) & \text{if } 1/2 \leq t \leq 1 \end{array} \right. \;=\; s_0.$$

Thus, we can apply the Universal Property of Quotient Space to get an induced continuous map $\bar{h} : \Sigma S^n \to S^{n+1}$:

$$\begin{tikzcd} S^n \times I \arrow[r, "h"] \arrow[d, "{[-]}"'] & S^{n+1} \\ \Sigma S^n \arrow[ur, dashed, bend right, "\bar{h}"'] & \end{tikzcd}$$

that moreover preserves the basepoints. Let us show that $\bar{h}$ is bijective. As h is surjective, we have $\mathrm{Im}(\bar{h}) = \mathrm{Im}(h)$. Moreover, as $(x, t) \mapsto 2tx + (1 - 2t)s_0$ and $(x, t) \mapsto 2(1 - t)x + (2t - 1)s_0$ are surjective from $S^n \times I$ to D^{n+1}, we have $\mathrm{Im}(h) = \mathrm{Im}(p_+) \cup \mathrm{Im}(p_-)$, and since $p_\pm$ is an homeomorphism, we have $\mathrm{Im}(p_\pm) = H^{n+1}_\pm$. That is why $\bar{h}$ is surjective:

$$\mathrm{Im}(\bar{h}) \;=\; \mathrm{Im}(h) \;=\; \mathrm{Im}(p_+) \cup \mathrm{Im}(p_-) \;=\; H^{n+1}_+ \cup H^{n+1}_- \;=\; S^{n+1}.$$

Now, we consider two different points $x \wedge t \neq x' \wedge t'$ in ΣX. We necessarily have $(x, t) \neq (x', t')$. On the one hand, we suppose neither $x \wedge t$ nor $x' \wedge t'$ is the basepoint $s_0 \wedge 0$. If $\bar{h}(x \wedge t)$ and $\bar{h}(x' \wedge t')$ are not in the same hemisphere, they are naturally different. Otherwise, if they are in the same one, let us say *w.l.o.g.* the upper one, then, as p_+ is injective and $xt + (1 - t)s_0 \neq x't' + (1 - t')s_0$, we have $h(x, t) \neq h(x', t')$, that is to say $\bar{h}(x \wedge t) \neq \bar{h}(x' \wedge t')$. On the other hand, we assume that one of the two points is the basepoint $s_0 \wedge 0$, let us say *w.l.o.g.* $x \wedge t = s_0 \wedge 0$. Then $\bar{h}(x \wedge t) = s_0$, and as $x' \wedge t' \neq x \wedge t = s_0 \wedge 0$, we have $(x', t') \notin (S^n \times \{0, 1\}) \cup (\{s_0\} \times I) = h^{-1}(s_0)$. So,

$$\bar{h}(x' \wedge t') = h(x', t') \neq s_0 = \bar{h}(x \wedge t).$$

That means $\bar{h}$ is injective, and thus bijective. Finally, we can see that the suspension ΣS^n is compact for it is the image of the compact $S^n \times I$ by the projection $[-]$ which is continuous. We also notice that S^{n+1} is Hausdorff. Hence, we have a continuous bijection $\bar{h} : \Sigma S^n \to S^{n+1}$ from a compact to a Hausdorff space, which implies $\bar{h}$ is a homeomorphism and: $\Sigma S^n \cong S^{n+1}$. $\square$

Proposition 1.4.20. The reduced cone and suspension define respectively the functors $C : \mathsf{Top}_* \to \mathsf{Top}_*$ and $\Sigma : \mathsf{Top}_* \to \mathsf{Top}_*$.

Proof. Recall from (1.18) that CX is homeomorphic to $X \times I/(X \times \{1\}) \cup (\{x_0\} \times I)$. We have:

$$C : \left\{ \begin{array}{ccc} Ob(\mathit{Top}_*) & \to & Ob(\mathit{Top}_*), \\ (X, x_0) & \mapsto & (CX, [(x_0, 0)]). \end{array} \right.$$

For any continuous map $f : (X, x_0) \mapsto (Y, y_0)$ in Top_*, we want to define $C(f) : CX \to CY$. First of all, we notice the following diagram commutes:

$$\begin{tikzcd} (X \times \{1\}) \cup (\{x_0\} \times I) \arrow[r, "x \mapsto *"] \arrow[d, hook, "i'"'] & \{*\} \arrow[d, "* \mapsto {[(y_0,\, 0)]}"] \\ X \times I \arrow[r, "{[-] \circ (f \times id)}"'] & CY \end{tikzcd}$$

because $[-] \circ (f \times id) \circ i((X \times \{1\}) \cup (\{x_0\} \times I)) = [(Y \times \{1\}) \cup (\{x_0\} \times I)] = \{[(y_0, 0)]\}$. Then, with the universal property of the pushout CX, we get the wanted map $C(f) : CX \to CY$:

$$\begin{tikzcd} (X \times \{1\}) \cup (\{x_0\} \times I) \arrow[r] \arrow[d, hook] & \{*\} \arrow[d, "g'"] \arrow[ddr, bend left] & \\ X \times I \arrow[r, "{[-]}"'] \arrow[drr, bend right, "{[-] \circ (f \times id)}"'] & CX \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "C(f)"'] & \\ & & CY \end{tikzcd}$$

which is more precisely:

$$C(f) : [(x,\, t)] \mapsto [(f(x),\, t)].$$

Now, if we have another continuous map $g : (Y,\, y_0) \to (Z,\, z_0))$ in Top_*, we have $C(g \circ f) = C(g) \circ C(f)$:

$$\forall [(x,\, t)] \in CX, \qquad C(g \circ f)([(x,\, t)]) \;=\; [(g \circ f(x),\, t)] \;=\; C(g)([(f(x),\, t)]) \;=\; C(g) \circ C(f)([(x,\, t)]).$$

Moreover, we also have $C(id) = id$:

$$\forall [(x,\, t)] \in CX, \qquad C(id)([(x,\, t)]) \;=\; [(id(x),\, t)] \;=\; [(x,\, t)] \;=\; id([(x,\, t)]).$$

Thus C is a functor, and we can do a similar reasonings to show that Σ is equally a functor. $\square$

Remark 1.4.21. We have constructed the natural transformations $id_{\mathit{Top}_*} \Rightarrow C$, $id_{\mathit{Top}_*} \Rightarrow \Sigma$ and $C \Rightarrow \Sigma$. Indeed, the "faces" of the following "cube" give the wanted natural transformations:

$$\begin{tikzcd}
X \arrow[rr] \arrow[dr] \arrow[dd] & & CX \arrow[dr] \arrow[dd] & \\
& Y \arrow[rr] \arrow[dd] & & CY \arrow[dd] \\
CX \arrow[rr] \arrow[dr] & & \Sigma X \arrow[dr] & \\
& CY \arrow[rr] & & \Sigma Y.
\end{tikzcd}$$

Proposition 1.4.22. The functor $\Sigma : \mathsf{Top}_* \to \mathsf{Top}_*$ is homotopically well-behaved, in the sense that:

$$f \simeq_* g \in Mor(\mathsf{Top}_*) \qquad \Rightarrow \qquad \Sigma f \simeq_* \Sigma g.$$

In other words, if two pointed spaces X and Y are homotopy equivalent, then the suspensions ΣX and ΣY are also homotopy equivalent.

Proof. Consider two continuous maps $f \simeq_* g : X \to Y$ in Top_* and denote $H : X \times I \to Y$ the corresponding homotopy. We recall from the proof of 1.4.20 that $\Sigma f : x \wedge t \to f(x) \wedge t$. Then, the map $x \wedge t \to H(x,\, t) \wedge t$ gives the wanted relation $\Sigma f \simeq_* \Sigma g$. $\square$

3. Computation of $H_n(S^m)$

Let $H_*(-;\, A)$ be an ordinary homology theory with an abelian group A. Consider a pointed space $(X,\, x_0)$ in Top_*. Let us compute $H_*(\Sigma X;\, A) := H_*((\Sigma X,\, \emptyset),\, A)$ in terms of $H_*(X;\, A) := H_*((X,\, \emptyset),\, A)$.

Lemma 1.4.23. Denote the natural projection $\pi : X \times I \to \Sigma X$, and the subspaces $U := \pi(X \times [0,\, 1))$ $V := \pi(X \times (0,\, 1])$ of ΣX. Then $(\Sigma X;\, U,\, V)$ is an excisive triad, where moreover U and V are open and contractible, and their intersection verifies $U \cap V \simeq X$.

Proof. In $I := [0,\, 1]$, the interval $[0,\, 1)$ is open, so $X \times [0,\, 1)$ is open in $X \times I$ together with the product topology. Then, as the suspension ΣX is endowed with the quotient topology, *i.e.* the finest topology that allows the natural projection $\pi : X \times I \to \Sigma X$ to be continuous, the set $U := \pi(X \times [0,\, 1))$ is open in ΣX. Similarly, we can define the set $V := \pi(X \times (0,\, 1])$ that is open in ΣX. We notice that $\mathring{U} \cup \mathring{V} = U \cup V = \Sigma X$, which implies $(\Sigma X;\, U,\, V)$ is an excisive triad. To show that U (resp. V) is contractible, we can do exactly as in the proof 1.4.18 that shows CX is contractible: we just need to replace I by $[0,\, 1)$ (resp. by $(0,\, 1]$) and CX by U (resp. by V). One can also demonstrate it seeing that U and V are both homotopic to CX, which is homotopic to a unit set since it is contractible. Now, let us verify $U \cap V \simeq X$. First of all, one can notice that $U \cap V \subset \Sigma X$ is homeomorphic to $X \times (0,\, 1)/_{\{x_0\} \times (0,\, 1)}$. We then define the following continuous maps:

$$f : \left\{ \begin{array}{ccc} X & \to & U \cap V, \\ x & \mapsto & x \wedge (1/2), \end{array} \right. \qquad \text{and} \qquad g : \left\{ \begin{array}{ccc} U \cap V & \to & X, \\ x \wedge t & \mapsto & x. \end{array} \right.$$

Now, we have $g \circ f \simeq id_X$ for we even have the equality, and the relation $f \circ g \simeq id_{U \cap V}$ is given by the homotopy $H : (U \cap V) \times I \to U \cap V$ that maps $(x \wedge t, s)$ to $x \wedge ((t - 1/2)s + 1/2)$. Thus, we deduce $U \cap V$ and X are homotopic. $\square$

Proposition 1.4.24. For any integer $n \geq 1$,

$$H_n(X;\, A) \cong H_{n+1}(\Sigma X;\, A).$$

Proof. The Mayer-Vietoris sequence (see the theorem 1.4.9) applied to the excisive triad $(\Sigma X;\, U,\, V)$ given by the lemma 1.4.23 is:

$$\dots \longrightarrow H_{n+1}(U) \oplus H_{n+1}(V) \longrightarrow H_{n+1}(\Sigma X) \longrightarrow H_n(U \cap V) \longrightarrow H_n(U) \oplus H_n(V) \longrightarrow \dots \quad (1.19)$$

Let $n \geq 1$ be an integer. Knowing that U is contractible and H_n is homotopy invariant, we have $H_n(U) \cong H_n(\{*\}) = 0$ using the dimension axiom (H5); and similarly $H_n(V) = 0$. Moreover, as $U \cap V \simeq X$, we have $H_n(U \cap V) \cong H_n(X)$ and it yields with proposition 1.3.5 the exact sequence (1.19) becomes:

$$0 \longrightarrow H_{n+1}(\Sigma X) \longrightarrow H_n(X) \longrightarrow 0$$

which implies $H_{n+1}(\Sigma X) \cong H_n(X)$ with the proposition 1.3.3. □

Remark 1.4.25. In particular, it implies that $H_n(\Sigma X; A)$ does not depend on the choice of the basepoint x_0 in $(X,\, x_0)$ for $n \geq 2$.

Lemma 1.4.26. We have the following exact sequence:

$$0 \longrightarrow H_1(\Sigma X;\, A) \longrightarrow H_0(X;\, A) \longrightarrow A \oplus A \longrightarrow H_0(\Sigma X;\, A) \longrightarrow 0.$$

Proof. The Mayer-Vietoris sequence (see the theorem 1.4.9) applied to the excisive triad $(\Sigma X;\, U,\, V)$ given by the lemma 1.4.23 gives the wanted exact sequence:

$$H_1(U) \oplus H_1(V) \longrightarrow H_1(\Sigma X) \longrightarrow H_0(U \cap V) \longrightarrow H_0(U) \oplus H_0(V) \longrightarrow H_0(\Sigma X) \longrightarrow H_{-1}(U \cap V),$$

because we have again $H_1(U) \oplus H_1(V) = 0$ and $H_0(U \cap V) \cong H_0(X)$, moreover both $H_0(U)$ and $H_0(V)$ are equal to $H_0(\{*\};\, A) = A$, and also we have $H_{-1}(U \cap V) \cong H_{-1}(X) = 0$. □

Proposition 1.4.27. For any integers $n \geq 0$ and $m \geq 0$ we have:

$$H_n(S^m;\, A) \cong \begin{cases} A \oplus A & \text{if } m = n = 0, \\ A & \text{if } m = n > 0 \text{ or } m > n = 0, \\ 0 & \text{otherwise.} \end{cases} \quad (1.20)$$

Proof. We split the proof into a few steps.

(I) Let us compute $H_n(S^0;\, A)$ for all integer n. By definition, the set S^0 contains only two points, so it is isomorphic to the coproduct of two unit sets: $S^0 \cong \{*\} \coprod \{*\}$. Then for any integer n:

$$H_n(S^0;\, A) \cong H_n\left(\{*\} \coprod \{*\};\, A\right) \overset{(\text{H3})}{\cong} H_n(\{*\};\, A) \oplus H_n(\{*\};\, A) \overset{(\text{H5})}{=} \begin{cases} A \oplus A & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

(II) Take open sets U and V exactly as in the lemma 1.4.23, in the particular case of $X = S^0$. So, as we saw in 1.4.19 that $\Sigma S^0 \cong S^1$, we have the excisive triad $(\Sigma S^0;\, U,\, V) \cong (S^1;\, U,\, V)$, where U and V are moreover contractible and verify $U \cap V \simeq S^0$.

(III) Let compute $H_n(S^1;\, A)$ for any integer $n \geq 0$. As $S^1 \cong \Sigma S^0$, we have the following isomorphisms for any integer $n \geq 2$:

$$H_n(S^1;\, A) \cong H_n(\Sigma S^0;\, A) \overset{1.4.24}{\cong} H_{n-1}(S^0;\, A) = 0,$$

Next, we want to compute $H_1(S^1;\, A)$ and $H_0(S^1;\, A)$. We apply the lemma 1.4.26 to $X := S^0$ and we get the following exact sequence:

$$0 \longrightarrow H_1(S^1;\, A) \longrightarrow H_0(S^0;\, A) \longrightarrow A \oplus A \longrightarrow H_0(S^1;\, A) \longrightarrow 0.$$

As we just saw, we have $H_0(S^0;\,A) \cong A \oplus A$, and we can give more precision about the morphisms using the notations of the Mayer-Vietoris sequence:

$$0 \xrightarrow{\phi_1} H_1(S^1;\,A) \xrightarrow{\Delta_1} A \oplus A \xrightarrow{\psi_0} A \oplus A \xrightarrow{\phi_0} H_0(S^1;\,A) \xrightarrow{\Delta_0} 0. \tag{1.21}$$

We can see Δ_1 is injective, so the group $H_1(S^1;\,A)$ is isomorphic to its image $\Delta_1(H_1(S^1;\,A)) \subseteq A \oplus A$. Moreover, as the above sequence (1.21) is exact, we have $\Delta_1(H_1(S^1;\,A)) \cong \ker(\psi_0)$. So we want to compute the kernel of $\psi_0 : (a,\,b) \mapsto (i_*(a,\,b),\, j_*(a,\,b))$, where $i : S^0 \hookrightarrow U$ and $j : S^0 \hookrightarrow V$. We claim that $i_* : (a,\,b) \mapsto a+b$. To show it, let us first denote W and E the two points of S^0, and $w : \{W\} \to S^0$ and $e : \{E\} \to S^0$ the associated inclusions. We have the pushout:

$$\begin{tikzcd}
0 \arrow[r, hook] \arrow[d, hook] & H_0(\{W\}) \arrow[d, "b \mapsto (0,\,b)"] \arrow[ddr, bend left, "w_*"] & \\
H_0(\{E\}) \arrow[r, "a \mapsto (a,\,0)"'] \arrow[drr, bend right, "e_*"'] & H_0(\{E\}) \oplus H_0(\{W\}) \arrow[ul, phantom, "\ulcorner" very near start] \arrow[dr, dashed, "(\textsc{h}3)", "\cong"'] & \\
& & H_0(S^0).
\end{tikzcd}$$

Consequently, up to isomorphism, we have $e_*(a) = (a,\,0)$ and $w_*(b) = (0,\,b)$ for any $a \in H_1(\{E\};\,A)$ and $b \in H_1(\{W\};\,A)$. Now, we have the retraction $r : U \to \{E\}$ of $i \circ e$ as follows:

$$\begin{tikzcd}
\{E\} \arrow[r, hook, "e"] & S^0 \arrow[r, hook, "i"] & U \arrow[ll, dashed, bend right, "r\colon u \mapsto E"']
\end{tikzcd}$$

which is a homotopy equivalence: indeed, we saw that U is contractible, *i.e.* U is relative homotopic to $\{E\}$, so we can find a homotopy equivalence $U \to \{E\}$, which is exactly r as it is the only map between these spaces. In particular, we then have $(i \circ e) \circ r \simeq id_U$. Therefore, as a homotopy equivalence of pointed spaces is a relative weak homotopy equivalence (see 1.2.28), with the axiom (H4) we have that $r_* : H_1(U;\,A) \to H_1(\{E\};\,A)$ is an isomorphism, and as $H_1(-;\,A)$ is a homotopy invariant functor:

$$(i \circ e)_* \circ r_* \;=\; (i \circ e \circ r)_* \;=\; id_*.$$

Thus, up to isomorphism, the map $(i \circ e)_*$ is precisely the identity. Similarly, we can show the same result for $(i \circ w)_*$. It finally yields: $\forall a \in H_1(\{E\};\,A)$, $\forall b \in H_1(\{W\} :\, A)$,

$$i_*(a,\,b) \;=\; i_*((a,\,0) + (0,\,b)) \;=\; i_*(a,\,0) + i_*(0,\,b) \;=\; (i \circ e)_*(a) + (i \circ w)_*(b) \;=\; a + b.$$

In a same way, we have $j_* : (a,\,b) \mapsto a + b$. It follows the kernel of $\psi_0 : (a,\,b) \mapsto (a+b,\, a+b)$ is exactly:

$$\ker(\psi_0) \;=\; \{(a,\,b) \in A \oplus A \mid a + b = 0\} \;=\; \{(a,\,-a) \mid a \in A\} \;\cong\; A,$$

which shows $H_1(S^1;\,A) \cong A$. Finally, let us compute $H_0(S^1;\,A)$. As the sequence (1.21) is exact, we have:

$$\ker(\phi_0) \;=\; \mathrm{Im}(\psi_0) \;=\; \{(a+b,\, a+b) \mid a,\, b \in A\} \;=\; \{(c,\,c) \mid c \in A\} \;\cong\; A,$$

which implies with the first isomorphism theorem and the fact ϕ_0 is surjective that we therefore have:

$$H_0(S^1,\,A) \;=\; \mathrm{Im}(\phi_0) \;\cong\; (A \oplus A)/\ker(\phi_0) \;\cong\; (A \oplus A)/A \;\cong\; A.$$

(IV) Due to 1.4.19 we have the relation $\Sigma S^n \cong S^{n+1}$, with enables to generalize the previous two steps (II) and (III) to any dimension. □

— Part 2 —

CW-complexes and Cofibrations

I. CW-complexes

In this section, we want to define and understand a new class of topological spaces, known as CW-complexes. We will see they are well-behaved spaces and that, up to weak homotopy equivalence, every space is a CW-complex: this is the cellular approximation. Well-known results were obtained due to J.H.C Whitehead, such as his theorem which we will see at the end of this section. For more details, see the references [AGP02, 5.1] and [Hat01, appendix].

1. Generalities

Definitions 2.1.1. *In Top, a topological space X is a* ***CW-complex*** *if there exists a sequence $X_0 \subseteq X_1 \subseteq X_2 \subseteq \ldots$ of topological spaces called a* ***CW-decomposition*** *of X, such that:*

(i) The first set X_0 is nonempty and is endowed with the discrete topology (*i.e.* every subset is open).

(ii) For any integer $n \geq 0$, we can find an index set J_n and a collection $\{\varphi_j : S^n \xrightarrow{\mathcal{C}^0} X_n\}_{j \in J_n}$ of continuous maps known as **attaching maps**, such that we have the pushout:

$$\begin{tikzcd} \coprod_{j \in J_n} S^n \arrow[r, "\sum_j \varphi_j"] \arrow[d, hook] & X_n \arrow[d] \\ \coprod_{j \in J_n} D^{n+1} \arrow[r] & X_{n+1} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \tag{2.1}$$

where S^n and D^{n+1} are given together with the induced topology of the usual topology on $\mathbb{R}^{n+1}$.

(iii) The union $\bigcup_{n \in \mathbb{N}} X_n$ together with the weak topology (namely, a set C is closed *if, and only if,* the intersection $C \cap X_n$ is closed for any $n \in \mathbb{N}$) is homeomorphic to X.

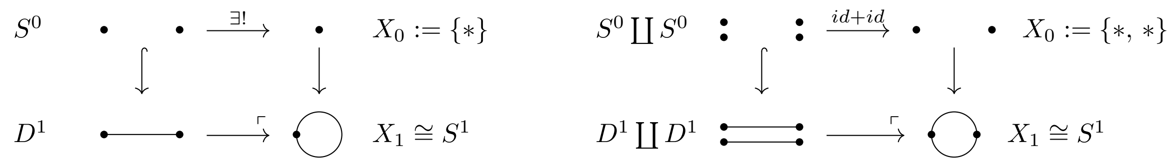
The points of X_0 are called **vertices** (or **vertex** in the singular), and a subspace X_n is called n**-skeleton** ($n \in \mathbb{N}$). In the pushout in (ii), the map $\coprod_{j \in J_n} D^{n+1} \to X_{n+1}$ induces maps $\phi_j^{n+1} : D_j^{n+1} \to X_{n+1}$ where we denote D_j^{n+1} a copy of D^{n+1} for $j \in J_n$. The restriction $\mathring{D}_j^{n+1} \to X_{n+1}$ of this induced map ϕ_j^{n+1} to the open $(n+1)$-disk $\mathring{D}_j^{n+1}$ is called **open** $(n+1)$**-cell**.

Remark 2.1.2. This restriction $\mathring{D}_j^{n+1} \to X_{n+1}$ is injective since each element x of $\mathring{D}_j^{n+1}$ is sent to its associated classe $[x]$ in the pushout X_{n+1}, and when we compute the quotient space, no part of the interior of any disk D_j^{n+1} is identified in X_{n+1} (there is identification only for the points of the boundary S^n). It shows an open $(n+1)$-cell is an homeomorphism from a space $\mathring{D}^{n+1}$ to its image. Thus, the image denoted $e_j^{n+1} \subseteq X_{n+1}$ of $\mathring{D}^{n+1}$ by an open $(n+1)$-cell is homeomorphic to $\mathring{D}^{n+1}$ and deserves to be also called **open** $(n+1)$**-cell**. Visually, an open $(n+1)$-cell is a space of dimension $n+1$: for instance, a 1-cell looks like a line segment, a 2-cell like an open disk, a 3-cell an open ball, *et caetera*... It is also common to generalize the notion of cell to the 0-cells, which would be then any point of X_0, also know as vertex. Using this notation, we would then have:

$$X_0 \cong \bigsqcup_{j \in J_{-1}} e_j^0,$$

where J_{-1} is defined as any set isomorphic to X_0. From now, we will simply use **cell** to describe an open n-cell for an integer $n \geq 0$. Notice that one can also define **closed** $(n+1)$**-cell** as being exactly $\bar{e}_j^{n+1} := \mathrm{Im}(\phi_j^{n+1})$ for j in J_n, but in general it is not homeomorphic to D^{n+1}.

Remarks 2.1.3. Given a CW-complex X, a CW-decomposition is not unique. For instance, we can construct two different CW-decompositions of the circle $X := S^1$, one beginning with $\{*\}$ and another one with $\{*, *\}$:



where, in the first case, we "glue" together the two ends of the segment D^1, and in the other case, we "glue" the two segments together (see the remark ...). In both case, we take $X_n = S^1$ for $n \geq 1$, and we have constructed two different CW-decompositions of S^1. Besides, we can similarly see that any n-sphere S^n is a CW-complex. Moreover, if for one $n \in \mathbb{N}$ we have $X_n \cong X$, then we do not need to precise the following spaces of the CW-decomposition: they are all necessary homeomorphic to X since $X_n \cong X$ is included in each of them, and their union is included in a space homeomorphic to X. To get such a CW-decomposition, one can notice that taking $J_k = \emptyset$ implies $X_{k+1} = X_k \coprod_\emptyset \emptyset = X_k$ for any $k \in \mathbb{N}$.

Remark 2.1.4. We defined the notion of CW-complex in *Top*, but we can also consider it in *Top*$_*$: a **pointed CW-complex** is a CW-complex X together with a basepoint that is also exactly the basepoint of its 0-skeleton X_0 (so, in particular, it is the basepoint of all its skeletons). One can then consider the categories *CW* and *CW*$_*$: the objects of *CW* are the CW-complexes, and its morphisms are the continuous maps from a CW-complex to another one; and similar for *CW*$_*$, but with pointed CW-complexes and pointed map. Note that in CW_*, we will use the wedge product $\vee$ rather than the coproduct $\coprod$ so that we may attach the basepoints together.

Proposition 2.1.5. A CW-complex X is the disjoint union of all its cells:

$$X \cong \bigsqcup_{n \in \mathbb{N}} \bigsqcup_{j \in J_{n-1}} e_j^n.$$

Proof. Let $x \in X$. We want to show that x can found in one and only one cell. We know that the 0-cells have an empty intersection since they are unit sets; and for any $n \geq 1$ it is the same with the n-cells because the open disks $\mathring{D}_j^n$ are disjoint in $\coprod_j D_j^n$, and the identification in the pushout which builds X_n concerns only the boundaries of the disks D_j^n. Moreover, for any $n \in \mathbb{N}$, an n-cell is necessary disjoint with an $(n+1)$-cell due to the fact that, in the coproduct of X_n and $\coprod_j D_j^{n+1}$, the n-cells and the $\mathring{D}_j^{n+1}$ are disjoint for any $j \in J_{n-1}$, and it stays disjoint when we quotient to get X_{n+1}, because, again, we identify only the boundaries of the disks D_j^{n+1} to points of X_n. This implies more generally that any two different cells are necessary disjoint. Now, if x is in X_0, then it belongs naturally to the cell $\{x\}$. Else, there is an integer $n \geq 1$ such that:

$$x \in X_n := X_{n-1} \coprod_{(\coprod_j S^{n-1})} \coprod_{j \in J_{n-1}} D_j^n.$$

We can assume *w.l.o.g.* by induction that x is not in X_{n-1}. So x is in $\coprod_j D_j^n$, and in particular one can find a $j \in J_{n-1}$ such that $x \in D_j^n$. Knowing that the boundary of D_j^n is identified to some point in X_{n-1}, we then have that x is not in this boundary, *i.e.* it is in the interior $\mathring{D}_j^n$, which is an n-cell. □

Proposition 2.1.6. Another possible decomposition of the n-sphere S^n as a CW-complex has one 0-cell and one n-cell. In fact, this particular decomposition is unique up to homeomorphism.

Proof. Let $n \geq 0$ be an integer. We can take $X_0 = \cdots = X_{n-1} = \{*\}$ (see the end of the previous remark), and then X_n as follows:

$$\begin{tikzcd} S^{n-1} \arrow[r, "id+id"] \arrow[d, hook] & X_{n-1} = \{*\} \arrow[d] \\ D^n \arrow[r] & X_n \cong S^n. \end{tikzcd}$$

□

The CW-complexes have some remarkable properties. Let us see some of them.

Proposition 2.1.7. Let X be a CW-complex. Then the following holds:

(i) X is locally path-connected.

(ii) If X is connected, then it is path-connected.

(iii) X is a T_1-space, that is every singleton $\{x\}$ of X is a closed space.

(iv) X is a normal space, thus Hausdorff.

Proof.

(i) To say that X is locally path-connected means that, given a point $x \in X$ and a neighborhood U of x in X, we can find a neighborhood $V \subset U$ of x that is path-connected. We have that X_0 is locally path-connected because it is discret: any singleton $\{x\}$ is open in X_0, so it is a neighborhood of x in X, it is naturally included in any neighborhood U of x, and it is abviously path-connected. Moreover, for any integer $n \geq 0$, the disk D^{n+1} is path-connected, then the coproduct $\coprod_{j \in J_n} D^{n+1}$ is locally path-connected, and so is X_{n+1} inductively, since attaching spaces preserves the property of being locally path-connected. Now, as union of locally path-connected spaces, we deduce that X is also locally path-connected.

(ii) It comes from a general result: any locally path-connected space X that is connected is in particular path-connected. Then, we conclude using the first point (i). We suggest showing this general result. We consider a point $x \in X$, and the set $U := \{y \in X \text{ connected to } x\}$ that is nonempty since x trivially belongs to it. Moreover, it is open: indeed, for any point $y \in U$, as X is locally path-connected, we can find a neighborhood $V \subseteq X$ of y that is path-connected; and this neighborhood V is included in U for any of its points can be connected to x "passing through" y. Now, let us consider $U' := X - U$ that is similarly open. *Reductio ad absurdum*, if we had $U' \neq \emptyset$, then we would have the contradiction that the connected space X is the union of two disjoint nonempty open sets. We deduct U' is empty, that is to say $U = X$, and X is path-connected.

(iii) We know that X_0 is a T_1-space since it is discret. For any $n \in \mathbb{N}$, the $(n+1)$-disk and then coproduct of $(n+1)$-disks are T_1-spaces. So, as attaching spaces preserves the property of being a T_1 space, we have by induction that X^{n+1} is also a T_1-space. Thus, as union of T_1-spaces, we conclude X is also locally a T_1-space.

(iv) As a recall, a space X is said normal if, given any disjoint closed sets A and B, there are neighborhoods U of A and V of B that are also disjoint. Using the Urysohn's Lemma, there is an equivalent definition: for any disjoint nonempty closed subsets A and B of X, we can find a continuous fonction $f : X \to I := [0, 1]$ such that:

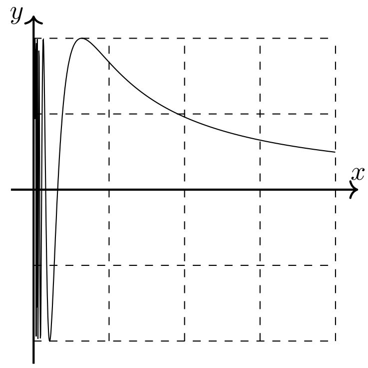
$$\forall x \in X, \qquad f(x) = \begin{cases} 0 & \text{if } x \in A, \\ 1 & \text{if } x \in B. \end{cases}$$

(see the reference "Urysohn's Lemma", J. Simon, 2007). Again using properties of attaching spaces and induction, we have that X_n is normal for all $n \in \mathbb{N}$. Let $A, B \subseteq X$ be disjoint closed sets. Then, there is continuous map $f_0 : X_0 \to I$ such that $f_0(x) = 0$ for all $x \in A \cap X_0$, and $f_0(x) = 1$ for all $x \in B \cap X_0$. Assume by induction we have already constructed a continuous map $f_{n-1} : X_{n-1} \to I$ for an integer $n \geq 1$ with $f_{n-1}(x) = 0$ for all $x \in A \cap X_{n-1}$ and $f_{n-1}(x) = 1$ for all $x \in B \cap X_{n-1}$, such that moreover $f_{n-1}|_{X_{n-2}} = f_{n-2}$ if $n \geq 2$. Consider $F := (A \cap X_n) \cup X_{n-1} \cup (B \cap X_n)$ and define $g_n : F \to I$ as follows:

$$\forall x \in F, \qquad g_n(x) := \begin{cases} 0 & \text{if } x \in A \cap X_n, \\ f_{n-1}(x) & \text{if } x \in X_{n-1}, \\ 1 & \text{if } x \in B \cap X_n. \end{cases}$$

Since X_n is normal and F is closed in X_n, we can extend g_n to a map $f_n : X_n \to I$ that satisfies the properties of g_n given just above. Now, one can define $f : X \to I$ such that $f|_{X_n} = f_n$ for any $n \in \mathbb{N}$, and it is continuous because X has the topology of the union. Hence, the CW-complex X is normal, and being also T_1 as seen in the previous point (iii), it is in particular a Hausdorff space. $\square$

Remark 2.1.8. The second point (ii) is a useful tool to show that a given topological space is not a CW-complexe simply by proving it connected but not path-connected. For instance, in $\mathbb{R}^2$ together with the usual topology, the famous graph of the fonction $x \mapsto \sin(\frac{1}{x})$ is connected, but not path-connected:



So it cannot be a CW-complex.

2. Subcomplex and n-equivalence

Definitions 2.1.9. *If X is a CW-complex and $A \subseteq X$ is a subspace, then we say that A is **subcomplex** of X if for every cell e_j^n satisfy the implication $A \cap e_j^n \neq \emptyset \Rightarrow \bar{e}_j^n \subseteq A$. We then call **CW-pair** the pair $(X,\, A)$ of spaces.*

Example. Every k-skeleton X_k of a CW-complex X is subcomplex. To show it, we consider a cell e_j^n such that $X_k \cap e_j^n \neq \emptyset$. We cannot have $n > k$ because a point in $X_k \subseteq X_{n-1}$ would be identified either to nothing else, or to a point of the boundary S^{n-1} of D^n, but not to any point of $\mathring{D}^n$. So we have $n \leq k$, and by definition of closed cell we have $\bar{e}_j^n \subseteq X_n \subseteq X_k$. This implies $(X,\, X_k)$ is a CW-pair for any $k \in \mathbb{N}$.

Proposition 2.1.10. Suppose X is a CW-complex and $K \subseteq X$ is compact. Then $K \subseteq X_n$ for some $n \in \mathbb{N}$. More specifically, we have $K \subseteq Y$ for a subcomplex $Y \subseteq X$, where Y has only a finite number of cells.

Proof. The compact K is well-defined because we saw in 2.1.7 that a CW-complex is Hausdorff. Due to the proposition 2.1.5, the disjoint union:

$$\bigcup_{n \in \mathbb{N}} \bigcup_{j \in J_{n-1}} e_j^n,$$

is an open cover of the compact K. We can then extract a finite subcover made up of cells. Since each cell is contained in a skeleton, it implies K is contained in a finite union of skeletons. Knowing that the sequence $\{X_n\}_{n \in \mathbb{N}}$ of the skeletons increases, it yields K is included in a skeleton. For the second part, it is sufficient to see that the previous finite subcover of K made up of cells can be the wanted subcomplex Y since the open cells are disjoint. $\square$

Let us introduce the concepts of n-connectedness and n-equivalence, and see how close they are.

Definitions 2.1.11. *Let $n \geq 0$ be an integer.*

(i) *If $n \geq 1$, an n-**equivalence** is a continuous map $f : X \to Y$ in the category **Top** such that for any $x_0 \in X$ the induced morphism f_* in **Set** defined by:*

$$f_* : \begin{cases} \pi_q(X,\, x_0) & \to & \pi_q(Y,\, f(x_0)) \\ [h] & \mapsto & f_*([h]) := [f \circ h] \end{cases}$$

is a bijection for $0 \leq q \leq n-1$ and a surjection for $q = n$.

(ii) *We say that a pair $(X,\, A)$ of spaces in Top_{rel} is n-**connected** if for all path component X_ν of X we have $A \cap X_\nu \neq \emptyset$, and if $\pi_q(X,\, A) = 0$ for $1 \leq q \leq n$.*

Remark 2.1.12. A pair $(X,\, A)$ is 0-connected *if, and only if,* we have $A \cap X_\nu \neq \emptyset$ for all path component $X_\nu \subseteq X$.

We state the two following statements for their beauty. Find them in [AGP02, 5.1].

Proposition 2.1.13. A pair (X, A) of spaces in $\mathsf{Top}_{\mathsf{rel}}$ is n-connected if, and only if, the inclusion map $i : A \hookrightarrow X$ is an n-equivalence. □

Proposition 2.1.14. Let X be a CW-complex and X_n be its n-skeleton, with $n \in \mathbb{N}$. Then the pair (X, X_n) is n-connected, and consequently the inclusion map $i : X_n \hookrightarrow X$ is an n-equivalence. □

3. Theorems

Now, let us take a look at two very useful tools that provide some interesting properties about CW-complexes: the Whitehead theorem and the cellular approximation. Find them in [AGP02, 5.1.36, 5.1.38].

Theorem 2.1.15. J.H.C Whitehead. A weak homotopy equivalence between pairs of CW-complexes is a homotopy equivalence. □

Theorem 2.1.16. Cellular approximation. CW-complexes are functorial, i.e. for any continuous map $f : X \to Y$ in Top, there exists a continuous map $\tilde{f} : \tilde{X} \to \tilde{Y}$ in CW such that we have the following commutatives diagram with weak equivalences:

$$\begin{tikzcd} \tilde{X} \arrow[r, "\sim"] \arrow[d, "\tilde{f}"'] & X \arrow[d, "f"] \\ \tilde{Y} \arrow[r, "\sim"] & Y. \end{tikzcd}$$

We say that $\tilde{X}$ and $\tilde{f}$ are **CW-approximations** of respectively X and f. Moreover, if f is in Top_* rather than in Top, we then have the same result but with $\tilde{f}$ in CW_*. □

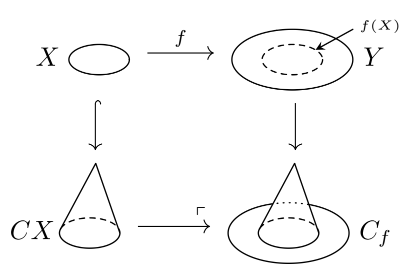
Remark 2.1.17. Combining the two previous theorem, we get that a CW-approximation is unique up to homotopy. Furthermore, one can show that, if $\tilde{X} \xrightarrow{\sim} X$ is a CW-approximation of a pointed space X, then $\Sigma\tilde{X} \xrightarrow{\sim} \Sigma X$ is also a CW-approximation. In other words, we have: $\widetilde{\Sigma X} \simeq \Sigma\tilde{X}$.

II. Cofiber sequence

1. Construction of the cofiber sequence

In order to be brief, we will mainly use geometric arguments in this section, based on the fact that a puhsout is a tool to "glue" objects together. We want to present the cofiber sequence and one of its applications: the long exact sequence of homotopy groups. First of all, let us introduce a new notion: mapping cones. Let $f : X \to Y$ be a continuous map in Top_*. See [AGP02, 3.3-5] for more details.

Definition 2.2.1. *The* ***mapping cone*** C_f *of* $f : X \to Y$ *is defined (up to isomorphism) as being the following pushout in* Top_**:*

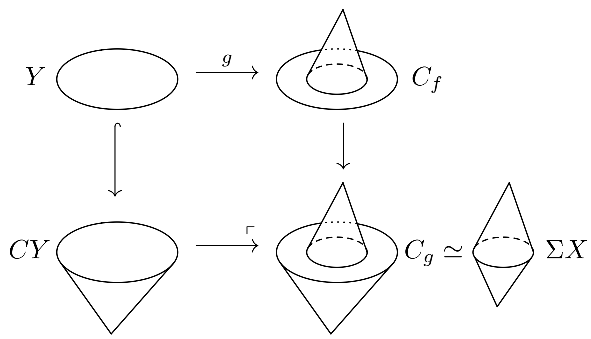


(2.2)

If we denote g the map $Y \to C_f$ in the previous diagram (which is more precisely the canonical projection from Y to C_f, see the proposition 1.1.27), we get the sequence:

$$X \xrightarrow{f} Y \xrightarrow{g} C_f. \tag{2.3}$$

Let us compute what would then be the mapping cone C_g of $g : Y \to C_f$:



(2.4)

So, up to homotopy, the mapping cone C_g of g is the suspension ΣX of X. Indeed, as we saw before, the reduced cone CY of Y is contractible, *i.e.* homotopic to a point; this means that, in order to compute the pushout C_g, we identify the whole image of g in C_f to one point. Now, denoting $h : C_f \to \Sigma X$ the induced continuous map in the previous diagram, we can complete the sequence (2.3) and we have:

$$X \xrightarrow{f} Y \xrightarrow{g} C_f \xrightarrow{h} \Sigma X.$$

Similarly, we can compute the mapping cone C_h of h, and we get the suspension ΣY of Y. Consequently, and more generally, we can build with mapping cones as before the following sequence in *Top*$_*$, that we call **cofiber sequence** of $f : X \to Y$, or also **Barratt-Puppe sequence** of $f : X \to Y$:

$$X \xrightarrow{f} Y \xrightarrow{g} C_f \xrightarrow{h} \Sigma X \xrightarrow{\Sigma f} \Sigma Y \xrightarrow{\Sigma g} \Sigma C_f \xrightarrow{\Sigma h} \Sigma^2 X \xrightarrow{\Sigma^2 f} \dots \qquad (2.5)$$

Remark 2.2.2. For some reasons, we can sometimes see in the literature $(-1)^n \Sigma^n f$ instead of simply $\Sigma^n f$. It is indeed due to the fact that, in a sense, this map takes the suspensions $\Sigma^n X$ and returns it to give $\Sigma^n Y$. However, up to isomorphism, we cannot see the difference between $(-1)^n \Sigma^n f$ and $\Sigma^n f$ (find some details on Hatcher, pages 297-299).

Proposition 2.2.3. The mapping cone is homotopically well-behaved, in that sense that, in Top$_*$, given any two continuous fonctions $f : X \to Y$ and $g : X \to Z$ such that $Y \simeq Z$, we have the equivalence $C_f \simeq C_g$.

Proof. Applying the universal property of the pushout C_j and the definition of the pushout C_f, we get a continuous map $\varphi : C_j \to C_f$ with the following commutative diagram:

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, hook] & Z \arrow[d, "{[-]}"] \arrow[dr, bend left, "\simeq"] & \\
X \times I \arrow[r, "{[-]}"'] \arrow[drr, bend right, "{[-]}"'] & C_g \arrow[dr, dashed, "\exists ! \varphi"] & Y \arrow[d, "{[-]}"] \\
& & C_f .
\end{tikzcd}$$

As this diagram commutes, one can learn some details about φ: in particular, it is a homotopy equivalence. $\square$

2. Some exact sequences in *Set*$_*$

We saw in the remark 1.2.14 that $[-, W]$ is a homotopy invariant contravariant functor for any relative space W. Now, to apply this contravariant functor to the Barratt-Puppe sequence (2.5) is interesting in the sense that we get a kind of "exact sequence" (in the category *Set*$_*$, not as previously in *Ab*). The objects of the category *Set*$_*$ are defined as sets containing a designated basepoint, and its morphisms are pointed maps of set.

Proposition 2.2.4. In Top_*, let W be any pointed space and $f : X \to Y$ a continuous map. We consider the kernel of f^* as being the following set:

$$\ker(f^*) := \left\{ [\varphi] \in [Y,\, W]_* \,\middle|\, f^*([\varphi]) := [\varphi \circ f] = [e_0] \right\},$$

where $e_0 : Y \to W$ is contant at the value given by the basepoint of W. Then, using this notion of kernel, the following sequence is exact in Set_*:

$$\ldots \xrightarrow{(\Sigma h)^*} [\Sigma C_f,\, W]_* \xrightarrow{(\Sigma g)^*} [\Sigma Y,\, W]_* \xrightarrow{(\Sigma f)^*} [\Sigma X,\, W]_* \xrightarrow{h^*} [C_f,\, W]_* \xrightarrow{g^*} [Y,\, W]_* \xrightarrow{f^*} [X,\, W]_*.$$

Remark 2.2.5. One can show that all the elements in this sequence are in fact groups, except for the three first on the right.

Proof. We will only show *w.l.o.g.* the exactness of the following part of the diagram:

$$[C_f,\, W]_* \xrightarrow{g^*} [Y,\, W]_* \xrightarrow{f^*} [X,\, W]_*$$

since it would be very similar for any other part. In other words, we want to prove the equality: $\mathrm{Im}(g^*) = \ker(f^*)$. On the one hand, let us consider $[\varphi]$ in $\mathrm{Im}(g^*)$. We can find $[\phi]$ in $[C_f,\, W]_*$ such that $g_\star([\phi]) = [\varphi]$. As C_f is the following pushout:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, hook, "i"'] & Y \arrow[d, "g"] \\ CX \arrow[r, "j"'] \arrow[ur, phantom, "\ulcorner", very near end] & C_f, \end{tikzcd}$$

we have by commutativity that $g \circ f = j \circ i$. Moreover, as the reduced cone CX is contractible (see ...), one can see that $j \circ i \simeq cst$ where $cst : X \to C_f$ is the constant map at the value given by the basepoint of C_f. So, as $[-,\, W]_*$ is homotopy invariant, we have $(j \circ i)^* = cst^*$ and then:

$$f^*([\varphi]) \;=\; f^*(g^*([\phi])) \;=\; (g \circ f)^*([\phi]) \;=\; (j \circ i)^*([\phi]) \;=\; cst^*([\phi]) \;=\; [\phi \circ cst] \;=\; [e_0],$$

which induces $\mathrm{Im}(g^*) \subseteq \ker(f^*)$. On the other hand, let $[\varphi]$ be in $\ker(f^*)$. It means $f^*([\varphi]) = [\varphi \circ f] = [e_0]$; in other words $\varphi \circ f$ and e_0 are homotopic. If we denote $(X_0,\, x_0) := X$ and $(W_0,\, w_0) := W$, one may consider $H : (X_0 \times I,\, \{x_0\} \times I) \to (W_0,\, w_0)$ a relative homotopy such that $H(-,\, 0) = \varphi \circ f$ and $H(-,\, 1) = e_0$. We notice that $H(x_0,\, t) = w_0$ and $H(x,\, 1) = w_0$ for any $(x,\, t) \in X \times I$, which implies $H(X \times \{1\} \cup \{x_0\} \times I) = \{w_0\}$. Thus, the Universal Property of Quotient Space yields there is a unique continuous map $\psi : CX \to W$ such that the following diagram commutes:

$$\begin{tikzcd} X \times I \arrow[r, "H"] \arrow[d, "{[-]}"'] & W \\ CX \arrow[ur, dashed, bend right, "\exists!\, \psi."'] & \end{tikzcd} \tag{2.6}$$

Then, we define the continuous map:

$$\psi' : \begin{cases} Y \vee CX & \to \quad W \\ x & \mapsto \begin{cases} \varphi(x) & \text{if } x \in Y, \\ \psi(x) & \text{if } x \in CX. \end{cases} \end{cases}$$

and we have for all $x \in X$:

$$\psi'(\underbrace{f(x)}_{\in Y}) = \varphi(f(x)) = H(x,\, 0) \overset{(2.6)}{=} \psi([(x,\, 0)]) = \psi(i(x)) = \psi'(\underbrace{i(x)}_{\in CX}).$$

It allows us to apply the Universal Property of Quotient Space that gives us the following commutative diagram:

$$\begin{tikzcd} Y \vee CX \arrow[r, "\psi'"] \arrow[d, "{[-]}"'] & W \\ C_f = Y \vee_X CX \arrow[ur, dashed, bend right, "\exists!\, \Psi"'] & \end{tikzcd} \tag{2.7}$$

where $[\Psi] \in [C_f, W]_*$. Now, one can observe that the image of $[\Psi]$ by g^* is exactly $[\varphi]$ since:

$$\forall y \in Y, \qquad \Psi \circ g(y) \; = \; \Psi([y]) \overset{(2.7)}{=} \psi'(y) \; = \; \varphi(y).$$

Hence, we have to reverse inclusion $\ker(f^*) \subseteq \mathrm{Im}(g^*)$, and we finally obtain the wanted equality. □

There exists also a relative version of the Barratt-Puppe sequence (2.5) of a continuous map $f : (X, A) \to (Y, B)$:

$$(X, A) \xrightarrow{f} (Y, B) \longrightarrow (C_f, C_{f|_{A\to B}}) \longrightarrow (\Sigma X, \Sigma A) \xrightarrow{\Sigma f} (\Sigma Y, \Sigma B) \longrightarrow ...$$

This sequence induces a very similar exact sequence to the one in the previous proposition.

Proposition 2.2.6. In $\mathsf{Top}_{\mathsf{rel}}$, let (W, C) be any relative space and $f : (X, A) \to (Y, B)$ a continuous map. We consider the kernel of f^* as being the following set:

$$\ker(f^*) := \left\{ [\varphi] \in [(Y, B), (W, C)]_* \,\middle|\, f^*([\varphi]) := [\varphi \circ f]) = [e_0] \right\},$$

where $e_0 : (Y, B) \to (W, C)$ is contant. Then the following sequence is exact in Set_*:

$$... \xrightarrow{(\Sigma f)^*} [(\Sigma X, \Sigma A), (W, C)] \longrightarrow [(C_f, C_{f|_{A\to B}}), (W, C)] \longrightarrow [(Y, B), (W, C)] \xrightarrow{f^*} [(X, A), (W, C)].$$

□

3. The long exact sequence of homotopy groups

We saw at the page 11 a definition of n-th homotopy group $\pi_n : \mathit{Top}_* \to \mathit{Set}$. Now, let us extend this definition to n-th reduced homotopy group $\pi_n : \mathit{Top}_{rel} \to \mathit{Set}$:

Definition 2.2.7. *For any integer $n \geq 1$, the n-**th reduced homotopy group** of a relative space (X, A) in Top_{rel} is the set:*

$$\pi_n(X, A) := [(D^n, S^{n-1}), (X, A)].$$

Remark 2.2.8. This is well an extension of the previous definition 1.2.16 since it coincides on Top_* for any integer n:

$$\pi_n(X, \{x_0\}) := [(D^n, S^{n-1}), (X, \{x_0\})] = [(D^n/S^{n-1}, \{*\}), (X, \{x_0\})] = [(S^n, *), (X, x_0)]_* = \pi_n(X, x_0).$$

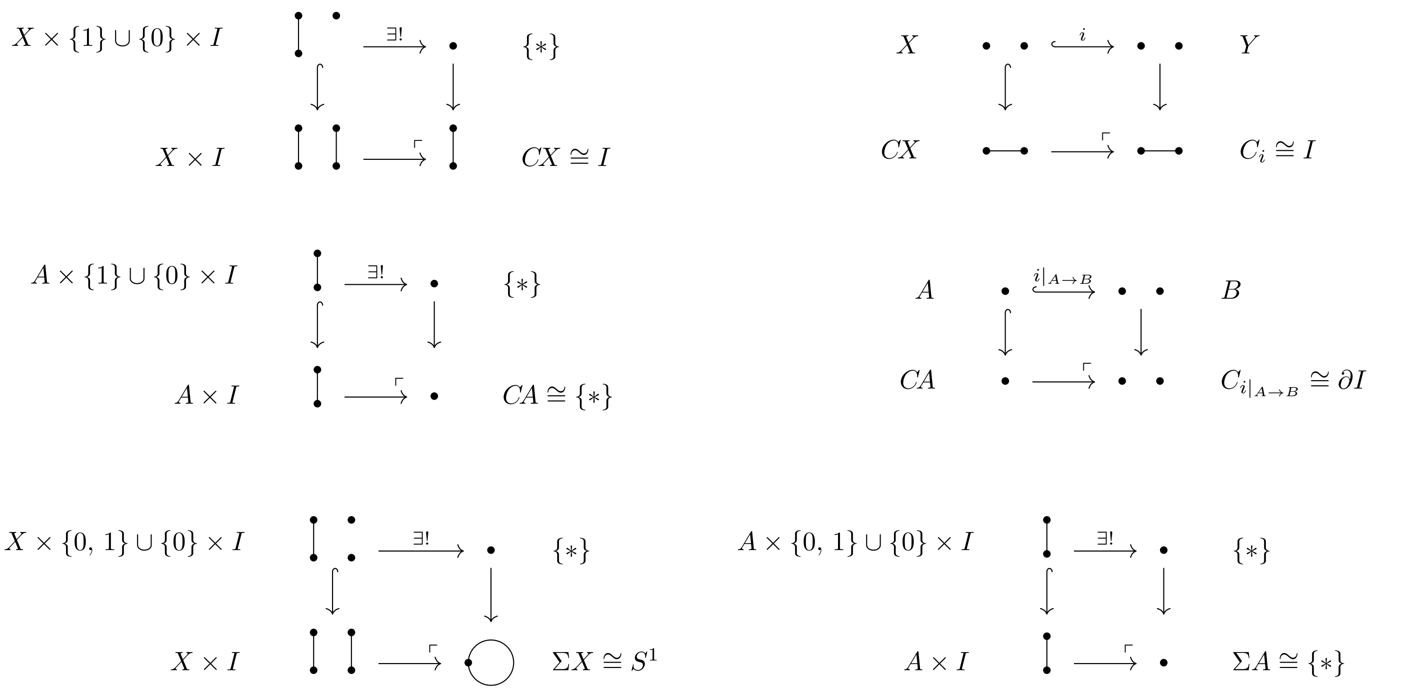
Now, taking f as being the inclusion $i : (\partial I, \{0\}) \hookrightarrow (\partial I, \partial I)$, we get the following interesting exact sequence:

Theorem 2.2.9. For any relative space (W, C), the following sequence is exact in Set_*:

$$... \longrightarrow \pi_2(W, C) \longrightarrow \pi_1(C, *) \xrightarrow{(\Sigma i)^*} \pi_1(W, *) \longrightarrow \pi_1(W, C) \longrightarrow \pi_0(C, *) \xrightarrow{i^*} \pi_0(W, *).$$

where $i : (\partial I, \{0\}) \hookrightarrow (\partial I, \partial I)$. This sequence is often called the **long exact sequence of homotopy groups**.

Proof. We denote $(X, A) := (\partial I, \{0\})$ and $(Y, B) := (\partial I, \partial I)$. First, compute the mapping cones C_i and $C_{i|_{A\to B}}$, and the suspensions ΣX and ΣA, with the following pushouts:



Now, one can clarify the elements of the exact sequence given by 2.2.6:

$$\begin{array}{lllllllll}
 & \ldots \xrightarrow{(\Sigma i)^*} & [(\Sigma X, \Sigma A), (W, C)] & \longrightarrow & [(C_i, C_{i|_{A \to B}}), (W, C)] & \longrightarrow & [(Y, B), (W, C)] & \xrightarrow{i^*} & [(X, A), (W, C)], \\
\textit{i.e.} & \ldots \xrightarrow{(\Sigma i)^*} & [(S^1, \{*\}), (W, C)] & \longrightarrow & [(I, \partial I), (W, C)] & \longrightarrow & [(\partial I, \partial I), (W, C)] & \xrightarrow{i^*} & [(\partial I, \{0\}), (W, C)], \\
\textit{i.e.} & \ldots \xrightarrow{(\Sigma i)^*} & [(S^1, \{*\}), (W, \{*\})] & \longrightarrow & [(D^1, S^0), (W, C)] & \longrightarrow & [(S^0, \{*\}), (C, \{*\})] & \xrightarrow{i^*} & [(S^0, \{*\}), (W, \{*\})], \\
\textit{i.e.} & \ldots \xrightarrow{(\Sigma i)^*} & \pi_1(W, *) & \longrightarrow & \pi_1(W, C) & \longrightarrow & \pi_0(C, *) & \xrightarrow{i^*} & \pi_0(W, *),
\end{array}$$

which is the wanted exact sequence. $\square$

III. Cofibrations

1. Generalities

Find details in [AGP02, 4.1-2].

Definition 2.3.1. *We say a relative space (X, A) has or satisfies the **HEP** (Homotopy Extension Property) if $A \subseteq X$ is closed, and if for any topological space Y and any continuous map $h : X \times \{0\} \cup A \times I \to Y$ the following diagram in Top commutes:*

$$\begin{tikzcd}
X \times \{0\} \cup A \times I \arrow[r, "h"] \arrow[d, hook] & Y \\
X \times I \arrow[ur, dashed, bend right, "\exists H."'] &
\end{tikzcd}$$

Definition 2.3.2. *A continuous map $j : A \to X$ in Top is a **cofibration**, and we denote $j : A \rightarrowtail X$, if:*

(i) *j is an embedding, i.e. $j : A \xrightarrow{\cong} \mathrm{Im}(j) \subseteq X$ is a homeomorphism onto its image,*

(ii) *The pair $(X, \mathrm{Im}(j))$ verify the HEP.*

Remarks 2.3.3. One can find in the literature one more condition in this definition: $\mathrm{Im}(j)$ must be closed; however, it is not necessary to precise it since it yields form (ii). Furthermore, as a cofibration $j : A \rightarrowtail X$ is an embedding, it is actually an inclusion up to isomorphism. This is why the first condition of the equivalence in the following proposition is realized non only for the cofibrations that are inclusions, but more generally for any cofibration up to isomorphism. Moreover, we will often write A instead of $\mathrm{Im}(j)$ since they are homeomorphic. In addition, note that one can define very similarly the notion of cofibration in Top_*.

> **Proposition 2.3.4.** Let $A \subseteq X$ be a closed subspace of $X \in Ob(\mathsf{Top})$. Then, the inclusion $j : A \hookrightarrow X$ is a cofibration if, and only if, the set $X \times \{0\} \cup A \times I$ is a retract of $X \times I$.

Proof. On the one hand, if $j : A \rightarrowtail X$ then (X, A) verifies the HEP, and in particular we have the following commutative diagram:

$$\begin{tikzcd} X \times \{0\} \cup A \times I \arrow[r, equal] \arrow[d, hook, "i"'] & X \times \{0\} \cup A \times I \\ X \times I \arrow[ur, dashed, bend right, "\exists r"'] & \end{tikzcd}$$

where $r \circ i = id$, *i.e.* r is a retract. On the other hand, let us assume that we have the retract $r : X \times I \to X \times \{0\} \cup A \times I$. Since j is an inclusion, it is an embedding, and $\mathrm{Im}(j)$ is closed as being homeomorphic to the closed space A. Moreover, for any any topological space Y and any continuous map $h : X \times \{0\} \cup A \times I \to Y$, the following diagram commutes:

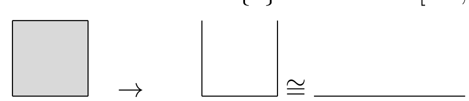
$$\begin{tikzcd} X \times \{0\} \cup A \times I \arrow[r, "h"] \arrow[d, hook, shift left, "i"] & Y \\ X \times I \arrow[u, bend left, "r"] \arrow[ur, dashed, bend right, "h \circ r"'] & \end{tikzcd}$$

because $(h \circ r) \circ i = h \circ (r \circ i) = h$. It yields $(X, \mathrm{Im}(j))$ verifies the HEP, and $j : A \rightarrowtail X$. $\square$

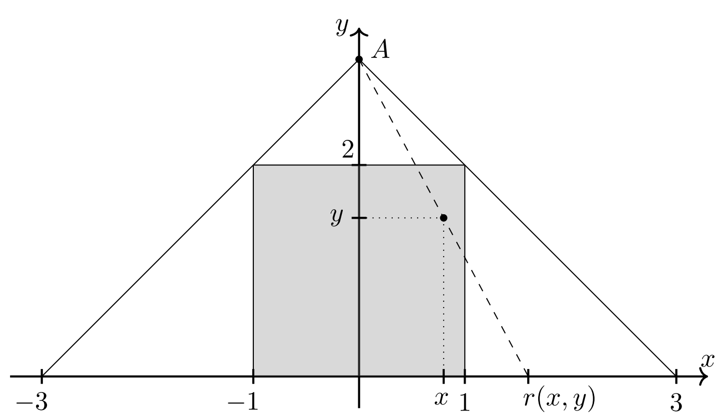
Examples.

(i) The inclusion $S^n \hookrightarrow D^{n+1}$ is cofibration for any $n \in \mathbb{N}$. Let us show this result for $n = 0$, the other cases being very similar. We work in $\mathbb{R}$ and $\mathbb{R}^2$ together with the usual topologies. To lighten the expressions, let us consider $D^1 \cong [-1, 1]$ and $I \cong [0, 2]$. We want to find a retraction r as follows:

$$r : \quad D^1 \times I \quad \to \quad D^1 \times \{0\} \cup S^0 \times I \cong [-3, 3]$$



We define the point $A := (0, 3) \in \mathbb{R}^2$. For any (x, y) in the square $D^1 \times I$, one can define $r(x, y)$ as being the intersection of $[-3, 3] \times \{0\}$ and the line passing through A and (x, y). In other words, applying the intercept theorem in the following drawing:



we get the expression $r(x, y) := \frac{3x}{3-y}$, which is continuous by composition of continuous maps. Moreover, if we denote i the inclusion $[-3, 3] \hookrightarrow D^1 \times I$ up to isomorphism, we well have $r \circ i = id_{[-3,\,3]}$, which means r is a retract and thus $S^0 \rightarrowtail D^1$ is a cofibration.

(ii) For any pointed space X, regarding X and its reduced cone CX as being in Top instead of Top_*, we have $X \rightarrowtail CX$. This result could be shown with the same idea as the previous one, using the projection passing by a wisely selected point. For more details, see the reference [Pic92, 2.3.8].

Proposition 2.3.5. Cofibrations are preserved under coproduct: given a set K and cofibrations $\{j_k : A_k \rightarrowtail X_k\}_{k\in K}$ in Top, we have the cofibration $j : A := \coprod_{k\in K} A_k \rightarrowtail X := \coprod_{k\in K} X_k$. Moreover, cofibrations are preserved under pushout, in the sens that given a pushout:

$$\begin{tikzcd} A \arrow[r] \arrow[d, tail, "j"'] & Y \arrow[d, "\tilde{j}"] \\ X \arrow[r] & P := X \coprod_A Y. \end{tikzcd} \tag{2.8}$$

implies $\tilde{j} : Y \rightarrowtail P$.

Proof. To show the first part of the proposition, we begin to see that $j : A \to X$ is an embedding since the $j_k : A_k \rightarrowtail X_k$ are so for $k \in K$. Let us then show that (X, A) verifies the HEP. To do so, after seeing that A (*i.e.* the image of j) is closed in X by coproduct of the closed spaces A_k for $k \in K$, consider a topological space Y and a continuous map $h : X \times \{0\} \cup A \times I \to Y$. We want to find a continuous map $H : X \times I \to Y$ such that its restriction on $X \times \{0\} \cup A \times I$ is precisely h. One can notice that $A \times I = (\coprod_k A_k) \times I \cong \coprod_k (A_k \times I)$, and the restriction $h|_{A\times I} : A \times I \to Y$ induces restrictions $h|_{A_k\times I} : A_k \times I \to Y$ for any $k \in K$. Similarly, for $k \in K$, we can define $h|_{X_k\times\{0\}} : X_k \times \{0\} \to Y$, and apply the HEP given by the cofibration j_k:

$$\begin{tikzcd} X_k \times \{0\} \cup A_k \times I \arrow[r, "h_j"] \arrow[d, hook] & Y \\ X_k \times I \arrow[ur, dashed, bend right, "\exists H_k,"'] & \end{tikzcd}$$

where $h_j := h|_{X_k\times\{0\}} + h|_{A_k\times I}$. Finally, we get the wanted continuous map H which for any $k \in K$ is defined as being H_k on A_k.

Now, let us consider the second part of the proposition. As j is a cofibration, it is in particular injective, which implies we do not identify two any different points in Y when we compute the quotient $P := X \coprod_A Y$. It yields j is injective and then an embedding. Next, let us show that (P, Y) verifies the HEP. The topology on the pushout P is the finest one such that the natural projection $[-]$ is continuous. With the proposition 1.1.27, we know that $\tilde{j}$ is actually exactly the projection $[-] : Y \to P$, so its image $\mathrm{Im}(\tilde{j}) \cong Y$ is closed in P. Now, consider a topological space Z and a continuous map $h : P \times \{0\} \cup Y \times I \to Z$ in *Top*. We want to find a continuous map $H : P \times I \to Z$ such that its restriction on $P \times \{0\} \cup A \times I$ is precisely h. We denote f and f' the respective maps $A \to Y$ and $X \to P$ in the diagram (2.8), and we get the following two continuous maps:

$$g : A \times I \xrightarrow{f+id} Y \times I \xrightarrow{h|_{Y\times I}} Z, \qquad \text{and} \qquad g' : X \times \{0\} \xrightarrow{f'+id} P \times \{0\} \xrightarrow{h|_{P\times\{0\}}} Z.$$

Therefore, applying the HEP induced by the cofibration $j : A \rightarrowtail X$, and then the Universal Property of Quotient Space, we have:

$$\begin{tikzcd} X \times \{0\} \cup A_k \times I \arrow[r, "g+g'"] \arrow[d, hook] & Z \\ X \times I \arrow[ur, dashed, bend right, "\exists \tilde{H},"'] & \end{tikzcd} \qquad \text{and then} \qquad \begin{tikzcd} (X \times I) \coprod (Y \times I) \cong (X \coprod Y) \times I \arrow[r, "\tilde{H}+h|_{Y\times I}"] \arrow[d, hook] & Z \\ (X \coprod_A Y) \times I \arrow[ur, dashed, bend right, "\exists H,"'] & \end{tikzcd}$$

where H is the wanted continuous map $P \times I \to Z$. Hence the result. □

Remark 2.3.6. As seen in the examples at the page 36, we have $X \rightarrowtail CX$ for any topological space X. So, for any continuous map $f : X \to Y$ in *Top*, we have $Y \rightarrowtail C_f$ since there is the pushout:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, tail] & Y \arrow[d, tail] \\ CX \arrow[r] & C_f. \end{tikzcd}$$

Proposition 2.3.7. If X is a CW-complex with a CW-decomposition $\{X_n\}_{n\in\mathbb{N}}$, then we have $X_n \rightarrowtail X_{n+1}$ for any integer $n \geq 0$.

Proof. Let $n \geq 0$ be an integer. We saw in the examples that $S^n \rightarrowtail D^{n+1}$, so the first part of the previous proposition implies we have the cofibration $\coprod_{j\in J_n} S^n \rightarrowtail \coprod_{j\in J_n} D^{n+1}$. Then as cofibrations are preserved under pushout we well have the wanted cofibration:

$$\begin{tikzcd}
\coprod_{j\in J_n} S^n \arrow[r] \arrow[d, tail] & X_n \arrow[d, tail] \\
\coprod_{j\in J_n} D^{n+1} \arrow[r] & X_{n+1}.
\end{tikzcd}$$

□

Remark 2.3.8. One can even show that $X_n \rightarrowtail X$ for any integer $n \geq 0$.

3. A relation between cofiber and cofibration

For some reasons, one can find in the literature the notion of **cofibers**, which are indeed mapping cones. It will make sense when we will see the proposition 2.3.12: given a cofibration $j : A \rightarrowtail X$, we have that the cone is actually the following quotient $C_j \simeq X/A$.

Proposition 2.3.9. Let $j : A \rightarrowtail X$ such that $A \simeq \{a_0\} \subseteq A$. Then $X \simeq X/A$.

Proof. We propose to show that the natural projection $q : X \to X/A$ is a homotopy equivalence. The fact that $A \simeq \{a_0\}$ means we can find a homotopy equivalence $f : A \to \{a_0\}$ and its homotopy inverse $g : \{a_0\} \to A$ such that in particular $id_A \simeq g \circ f$. It yields there exists a homotopy $H : A \times I \to A$ such that $H(-, 0) = id_A$ and $H(-, 1) = g \circ f(-) = g(a_0)$. Due to the cofibration $j : A \rightarrowtail X$, the pair (X, A) verifies the HEP, which means the following diagram commutes:

$$\begin{tikzcd}
X \times \{0\} \cup A \times I \arrow[r, "id_X + H"] \arrow[d, hook] & X \\
X \times I \arrow[ur, dashed, bend right, "\exists F"'] &
\end{tikzcd}$$

We define the map $F_t : X \to X$ as being $F_t : x \mapsto F(x, t)$ for any $t \in I$. More precisely we have $F_1|_A = H(-, 1) = g(a_0)$, so, by the Universal Property of Quotient Space, the map F_1 determines a map $q' : X/A \to X$ such that $q' \circ q = F_1$. Therefore, since $F_0 = F(-, 0) = id_X$, we notice that F induces $q' \circ q \simeq id_X$. Conversely, since $F_t(A) = H(A, t) \subseteq A$ for any $t \in I$, one can apply the Universal Property of Quotient Space to get the following commutative diagram:

$$\begin{tikzcd}
X \arrow[r, "q \circ F_t"] \arrow[d, "q + id"'] & X/A \\
(X/A) \times I \arrow[ur, dashed, bend right, "\exists G"'] &
\end{tikzcd}$$

In particular, for any $x \in X$ we have $G(q(x), 0) = q \circ F_0(x) = q \circ id_X(x) = q(x)$ and $G(q(x), 1) = q \circ F_1(x) = q \circ q'(q(x))$. Then, as the projection $q : X \to X/A$ is injective, it means G determines $id_{X/A} \simeq q \circ q'$, and thus q and q' are homotopy inverses. □

Lemma 2.3.10. In Top_*, let $A \subseteq X$ be a closed pointed space in a pointed space X, and let $f : A \to Y$ be a continuous map that lands in a pointed space Y. Recall we have the following pushout:

$$\begin{tikzcd}
A \arrow[r, hook] \arrow[d, "f"'] & X \arrow[d] \\
Y \arrow[r] & X \vee_A Y.
\end{tikzcd}$$

We then have the homeomorphism: $\quad (X \vee_A Y)/Y \cong X/A$.

Remark 2.3.11. The inclusion $A \hookrightarrow X$ is injective, so we do not identify together two different points in Y. That is why one can say $Y \subseteq X \vee_A Y$, and then compute the quotient $(X \vee_A Y)/Y$.

Proof. One the one hand, with the Universal Property of Quotient Space, we have:

$$\begin{tikzcd}
X \arrow[r, "{[-]}"] \arrow[d, "\text{projection}"', "\pi"] & X \vee_A Y \arrow[r, "q"] & (X \vee_A Y)/Y \\
X/A \arrow[urr, dashed, bend right, "\exists!\tilde{\alpha}"'] & &
\end{tikzcd} \tag{2.9}$$

where q is the natural projection from $X \vee_A Y$ to $(X \vee_A Y)/Y$. Indeed, we can apply this property because two different points in X are identified in X/A *if, and only if,* they are in A, and because $q \circ [-]$ is constant on A: any a and b in A are respectively identified to $f(a)$ and $f(b)$ in Y when we compute $X \vee_A Y$, and then taking the quotient by Y we have that $f(a)$ and $f(b)$ are identified together, *i.e.* a and b are so. On the other hand, if we define a constant map $j : Y \to \{*\} \hookrightarrow X/A$, the following diagram commutes:

$$\begin{tikzcd}
A \arrow[r, hook] \arrow[d, "f"'] & X \arrow[d, "\pi"] \\
Y \arrow[r, "j"'] & X/A
\end{tikzcd}
\quad \text{and then:} \quad
\begin{tikzcd}
A \arrow[r, hook] \arrow[d, "f"'] & X \arrow[d] \arrow[ddr, bend left, "\pi"] & \\
Y \arrow[r] \arrow[drr, bend right, "j"'] & X \vee_A Y \arrow[dr, dashed, "\exists!\beta"] & \\
& & X/A.
\end{tikzcd} \tag{2.10}$$

As the lower triangle commutes, it follows β is constant on $Y \subseteq X \vee_A Y$, and we can apply the Universal Property of Quotient Space:

$$\begin{tikzcd}
X \coprod_A Y \arrow[r, "\beta"] \arrow[d, "q"'] & X/A \\
(X \coprod_A Y)/Y \arrow[ur, dashed, "\exists!\tilde{\beta}."'] &
\end{tikzcd} \tag{2.11}$$

Now, let us show that $\tilde{\beta} \circ \tilde{\alpha} = id$ and $\tilde{\alpha} \circ \tilde{\beta} = id$. For any $\tilde{x} \in X/A$, as the projection π is surjective, we can find a $x \in X$ such that $\pi(x) = \tilde{x}$, and we have:

$$\tilde{\beta} \circ \tilde{\alpha}(\tilde{x}) \;=\; \tilde{\beta} \circ \tilde{\alpha}(\pi(x)) \;\overset{(2.9)}{=}\; \tilde{\beta} \circ q([x])) \;\overset{(2.11)}{=}\; \beta([x]) \;\overset{(2.10)}{=}\; \pi(x) \;=\; \tilde{x} \;=\; id(\tilde{x}).$$

Conversely, for any $\tilde{z} \in (X \vee_A Y)/Y$, as the projection q is surjective, we can find a $z \in X \vee_A Y$ such that $q(z) = \tilde{z}$. If we can write z as $[x]$ with $x \in X$, then we have:

$$\tilde{\alpha} \circ \tilde{\beta}(\tilde{z}) \;=\; \tilde{\alpha} \circ \tilde{\beta}(q([x])) \;\overset{(2.11)}{=}\; \tilde{\alpha} \circ \beta([x]) \;\overset{(2.10)}{=}\; \tilde{\alpha} \circ \pi(x) \;\overset{(2.9)}{=}\; q([x]) \;=\; \tilde{z} \;=\; id(\tilde{z}).$$

Else, we can found a $y \in Y$ such that $z = [y]$. One can assume *w.l.o.g.* that $A \neq \emptyset$ (because in the other case, the lemma is obviously true). So there is $a \in A$, and the element $f(a)$ in Y is identified to a in X when we compute $X \vee_A Y$. Next, in $(X \vee_A Y)/Y$, we identify $f(a) \in Y$ to any $y \in Y$. Therefore, all $y \in Y$ are identified to the element a of X when computing the quotient $(X \vee_A Y)/Y$, and we have $q([a]) = q([y]) = \tilde{z}$, *i.e.* we are again in the previous case where z could be written as $[x]$ with $x \in X$. Hence the two wanted equalities, and consequently the wanted homeomorphism. $\square$

Proposition 2.3.12. Given a cofibration $j : A \rightarrowtail X$ in Top_*, we have:

$$C_j \simeq X/A.$$

Proof. Due to the lemma and the previous proposition, we have the following commutatives diagram:

$$\begin{tikzcd}
A \arrow[r, tail, "j"] \arrow[d, tail] & X \arrow[r, "{[-]}"] \arrow[d, tail] & X/A \\
CA \arrow[r, tail] & C_j \arrow[r, "3.34.", "\simeq"'] & C_j/CA. \arrow[u, "3.35."', "\cong"]
\end{tikzcd}$$

which induces the desired result. $\square$

Remark 2.3.13. Then, up to homotopy, the cofiber sequence (2.5) of a cofibration $j : A \rightarrowtail X$ is:

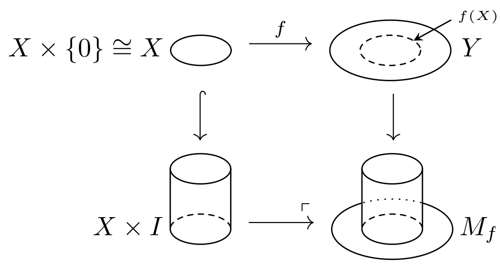
$$A \overset{j}{\rightarrowtail} X \rightarrowtail X/A \rightarrowtail \Sigma X \overset{\Sigma j}{\rightarrowtail} \Sigma Y \rightarrowtail \Sigma(X/A) \rightarrowtail \Sigma^2 X \overset{\Sigma^2 j}{\rightarrowtail} \Sigma^2 Y \rightarrowtail ...$$

where the cofibrations come from the remark 3.30. because this is a sequence of mapping cones.

4. Mapping cylinder

The following theorem tells us *grosso modo* that cofibrations are "everywhere" in *Top*$_*$, in the sense that any continuous map is a cofibration up to homotopy equivalence. But first let us define the mapping cylinder, which is a pushout that is very similar to the mapping cone:

Definition 2.3.14. *In Top$_*$, we call **mapping cylinder** of a continuous map $f : X \to Y$, and we denote M_f, the following pushout:*



(2.12)

Remark 2.3.15. More precisely, one can notice than we have $M_f \simeq Y \times \{0\} \cup f(X) \times I$ if f is injective.

Theorem 2.3.16. For any continuous map $f : X \to Y$ in Top$_*$, we can find a cofibration $j_f : X \rightarrowtail M_f$ and a homotopy equivalence $p_f : M_f \xrightarrow{\simeq} Y$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, tail, "j_f"'] & & Y \\ & M_f \arrow[ur, "p_f"', "\simeq"] & \end{tikzcd} \tag{2.13}$$

Proof. As being the pushout (2.12), the mapping cylinder verifies: $M_f \cong (X \times I) \vee_X Y$. We define the continuous maps j_f and p_f as follows:

$$j_f : \left\{ \begin{array}{ccc} X & \to & M_f \\ x & \mapsto & [(x,\, 1)], \end{array} \right. \quad \text{and} \quad p_f : \left\{ \begin{array}{ccc} M_f & \to & Y \\ [y] & \mapsto & y \\ [(x,\, s)] & \mapsto & f(x) \end{array} \right. \quad \text{defined by} \quad \begin{tikzcd} X \times \{0\} \cong X \arrow[r, "f"] \arrow[d, hook] & Y \arrow[d, "{[-]}"] \arrow[ddr, equal, bend left] & \\ X \times I \arrow[r, "{[-]}"'] \arrow[dr, bend right, "{(x,\, s) \mapsto x}"'] & M_f \arrow[dr, dashed, "\exists! p_f"] & \\ & X \arrow[r, "f"'] & Y. \end{tikzcd} \tag{2.14}$$

First, let us show that $j_f : X \to M_f$ is a cofibration. It is an embedding because its image is $\mathrm{Im}(j_f) = [X \times \{1\}]$, which is isomorphic to $X \times \{1\} \cong X$ in M_f since only points of the form $(x,\, 0) \in X \times I$ or $f(x) \in Y$ are identified in M_f. Now, up to isomorphism, one can say that $j_f : X \hookrightarrow M_f$ is an inclusion, and we want to apply the result 2.3.4: we then need to show that $M_f \times \{0\} \cup X \times I$ is a retract of $M_f \times I$. Define the map:

$$\tilde{r} : ((X \times I) \vee Y) \times I \to M_f \times \{0\} \cup X \times I$$

as follows:

$$\begin{array}{rrcl} \forall (y,\, t) \in Y \times I, & \tilde{r}(y,\, t) & := & ([y],\, 0) \\ \forall ((x,\, s),\, t) \in (X \times I) \times I, & \tilde{r}((x,\, s),\, t) & := & \left\{ \begin{array}{ll} \left(x,\, t - \frac{(1-s)(1-t)}{s}\right) & \text{if } s \geq 1-t \text{ and } s \neq 0 \\ \left([(x,\, s + \frac{st}{1-t})],\, 0\right) & \text{if } s \leq 1-t \text{ and } s \neq 0 \\ ([(x,\, 0)],\, 0) & \text{if } s = 0 \end{array} \right. \end{array}$$

which is continuous by composition of continuous maps. Then, one can observe that for any $[(x, 0)] = [y]$ in M_f we have:

$$\forall t \in I, \qquad \tilde{r}((x, 0), t) = ([(x, 0)], 0) = ([y], 0) = \tilde{r}(y, 0).$$

It yields we can apply the Universal Property of Quotient Space to get a continuous map $r : M_f \times I \to M_f \times \{0\} \cup X \times I$ that verifies in particular $r([z], t) = \tilde{r}(z, t)$ for any $z \in (X \times I) \vee Y$ and $t \in I$. Denoting by i the inclusion $M_f \times \{0\} \cup X \times I \hookrightarrow M_f \times I$, it follows:

$$\forall [z] \in M_f, \qquad r \circ i([z], 0) = r([z], 0) = \tilde{r}(z, 0) = ([z], 0) = id([z], 0),$$

that is to say r is the wanted retraction, and thus $j_f : X \rightarrowtail M_f$.
Next, let us show that p_f is homotopy equivalence. We denote $i' : Y \to M_f$ the projection $y \mapsto [y]$. In the diagram (2.14) that defines p_f, we can see that $p_f \circ i' = id_Y$. Then, in order to define a homotopy $H : M_f \times I \to M_f$ from $i' \circ p_f$ to id_{M_f}, we consider $H([y], t) = [y]$ for all $y \in Y$ and $H([(x, s)], t) = [(x, st)]$ for all $(x, s) \in X \times I$, both for any $t \in I$. This map H thereby constructed is continuous by composition of continuous, and is exactly the wanted homotopy since for any $y \in Y$ and any $(x, s) \in X \times I$ we have:

$$\begin{array}{cc} H([y], 0) = [y] = i'(y) = i' \circ p_f(y) & H([(x, s), 0) = [(x, 0)] \overset{(2.14)}{=} [f(x)] =' i(f(x)) = i' \circ p_f([(x, s)]), \\ H([y], 1) = [y] = id([y]) & H([(x, s), 1) = [(x, s)] = id([(x, s)]). \end{array}$$

Hence, since the digram (2.13) is commutative: $p_f \circ j_f(x) = p_f([(x, 1)]) = f(x)$ for any $x \in X$, we get the wanted result. $\square$

Remark 2.3.17. Even though $\tilde{r}$ does not seem to be well-define on $(X \times I) \times I$ when $s = 1 - t$ and $s \neq 0$ because the two expressions do not match, it is indeed secretly well-defined : the trick is just that we used X instead of $\mathrm{Im}(j_f)$ to lighten the notations, and it induces this difference of notation.

Remark 2.3.18. With the notation of the diagram (2.13), we have the homotopy equivalence $C_{j_f} \simeq C_f$ due to the proposition 2.2.3.

— Part 3 —

Axiomatic reduced homology theory and Singular homology

I. Axiomatic reduced homology theory

We saw previously the notion of homology theory. Now, we will define a very close concept: the reduced homology theory. We will also observe that there is a one-to-one relation between those two kinds of theories: it means we can choose the most convenient one depending on the context we work on. Basically, the main difference is that one is defined on Top_{rel} while the other one on Top_* only. Note that the Dold-Thom theorem 3.2.11 will insure the existence of such a reduced homology theory. Here, the main reference will be [May99].

Definition 3.1.1. *A **(generalized) reduced homotopy theory** $\tilde{E}_*$ is a family $\{\tilde{E}_n : \mathsf{Top}_* \to \mathsf{Ab} \mid n \in \mathbb{Z}\}$ of homotopy invariant functors that satisfies the following four axioms:*

(H̃1) ***Exactness:*** *for any continuous map $f : A \to X$ in Top_*, the following sequence is exact for any $n \in \mathbb{Z}$:*

$$\tilde{E}_n(A) \xrightarrow{\tilde{E}_n(f)} \tilde{E}_n(X) \xrightarrow{\tilde{E}_n([-])} \tilde{E}_n(C_f).$$

(H̃2) ***Suspension:*** *For any integer n and any pointed space X, there is a natural group isomorphism:*

$$\tilde{E}_n(\Sigma) : \tilde{E}_n(X) \xrightarrow{\cong} \tilde{E}_{n+1}(\Sigma X).$$

(H̃3) ***Additivity:*** *for any collection $\{X_j\}_{j\in J}$ of pointed spaces, the inclusions $i_{j_0} : X_{j_0} \hookrightarrow \bigvee_{j\in J} X_j$ for $j_0 \in J$ induce isomorphisms in Ab for all integer n:*

$$\sum_{j\in J} \tilde{E}_n(i_j) : \left\{ \begin{array}{ccc} \bigoplus_{j\in J} \tilde{E}_n(X_j) & \xrightarrow{\cong} & \tilde{E}_n\left(\bigvee_{j\in J} X_j\right) \\ \sum_{j\in J} x_j & \longmapsto & \sum_{j\in J} \tilde{E}_n(i_j)(x_j) \end{array} \right.$$

(H̃4) ***Invariance with weak equivalences:*** *If $f : X \to Y$ is a relative weak homotopy equivalence in Top_*, then the following homomorphism is an isomorphism in Ab for any integer n:*

$$\tilde{E}_n(f) : \tilde{E}_n(X) \xrightarrow{\cong} \tilde{E}_n(Y).$$

*We also say that the reduced homology theory $\tilde{E}_*$ is **ordinary** if moreover the following axiom is verified:*

(H̃5) ***Dimension:*** *there exists an abelian group A such that:*

$$\tilde{E}_n(S^0) = \left\{ \begin{array}{ll} A & \text{if } n = 0 \\ 0 & \text{otherwise,} \end{array} \right.$$

where the basepoint of S^0 is any of its two points.

Remark 3.1.2. One can extend the exact sequence given in (h̃1). Indeed, let $f : A \to X$ be a continuous map in Top_*, and n be an integer. The projection $g : X \to C_f$ given in the pushout (2.2) is continuous in Top_*, and applying then the exactness axiom (h̃1) to g, we get the exact sequence $\tilde{E}_n(X) \to \tilde{E}_n(C_f) \to \tilde{E}_n(C_g)$. Then, we have with the diagram (2.4) that $C_g \simeq \Sigma A$. So, the previous exact sequence is more precisely $\tilde{E}_n(X) \to \tilde{E}_n(C_f) \to \tilde{E}_n(\Sigma A)$. Now, one can observe with the suspension axiom (h̃2) that $\tilde{E}_n(\Sigma A) \cong \tilde{E}_{n-1}(A)$, and therefore we have the following exact sequence:

$$\tilde{E}_n(A) \longrightarrow \tilde{E}_n(X) \longrightarrow \tilde{E}_n(C_f) \longrightarrow \tilde{E}_{n-1}(A) \longrightarrow ...$$

which can be extended from $\tilde{E}_{n-1}(A)$ with the exactness axiom (h̃1) so forth and so on. Moreover, in the particular case of a cofibration $f : A \rightarrowtail X$, we have with the proposition 2.3.12. that $C_f \simeq X/A$, so the exact sequence becomes:

$$\tilde{E}_n(A) \longrightarrow \tilde{E}_n(X) \longrightarrow \tilde{E}_n(X/A) \longrightarrow \tilde{E}_{n-1}(A) \longrightarrow ...$$

since it is not a problem to work up to homotopy as the functors $\tilde{E}_n$ are homotopy invariant for $n \in \mathbb{Z}$.

Remark 3.1.3. One can prove that the exactness axiom (h̃1) is equivalent to another axiom (h̃1') defined as follows: for any cofibration $j : A \rightarrowtail X$ in Top_*, the following sequence is exact for any $n \in \mathbb{Z}$:

$$\tilde{E}_n(A) \xrightarrow{\tilde{E}_n(j)} \tilde{E}_n(X) \xrightarrow{\tilde{E}_n([-])} \tilde{E}_n(X/A).$$

Indeed, we showed (h̃1)⇒(h̃1') in the previous remark, and conversely we have (h̃1')⇒(h̃1) thanks to the theorem 2.3.16: cofibrations are "everywhere" in Top_*.

We will adopt for a reduced homotopy theory $\tilde{E}_*$ the same abuse of notation as the for a homotopy theory. In particular, we denote $\tilde{H}_*(-;\, A) := \tilde{E}_*(-)$ if E_* is ordinary with the given abelian group A.

Remark 3.1.4. Let $(X,\, x_0)$ be a pointed space in Top_*. We can always assume $\{x_0\} \hookrightarrow X$ to be a cofibration, up to weak equivalence.

The following proposition explains that, in order to understand a reduced or not homology theory, it is enough to study it on the CW-complexes.

> **Proposition 3.1.5.** Show that a homology theory E_* on $\mathsf{Top}_{\mathsf{rel}}$ determines and is determined by its restriction to a generalized homology theory E_* on pairs of CW-complexes, and similarly for a reduced homology theory $\tilde{E}_*$ on Top_* with its restriction to the based CW-complexes.

Proof. We know due to the cellular approximation that any topological space X is weakly equivalent to a CW-complex $\tilde{X}$. It yields the wanted result with the invariance by weak equivalence (h4) and (h̃4). ☐

Lemma 3.1.6. Let $n \in \mathbb{Z}$. Given any homology theory E_* and any topological space X, we have:

$$E_n(X) = E_n(\{*\}) \oplus E_n(X,\, *).$$

Proof. There is a retraction $r : X \to \{*\}$ of the inclusion $i : \{*\} \hookrightarrow X$, so using the exactness axiom (h1) we get the exact sequence:

$$\begin{tikzcd} ... \arrow[r] & E_n(\{*\}) \arrow[r, "E_n(i)"] & E_n(X) \arrow[l, bend left, dashed, "E_n(r)"] \arrow[r] & E_n(X,\, *) \arrow[r] & ... \end{tikzcd}$$

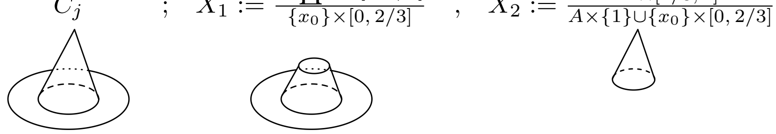
We notice that $E_n(r)$ is a retraction of $E_n(i)$ since $E_n(r) \circ E_n(i) = E_n(r \circ i) = E_n(id) = id$, and we can then adapt the splitting lemma (see page 16) to this context so that we exactly get the wanted isomorphism. ☐

Lemma 3.1.7. Let $j : A \rightarrowtail X$ be a cofibration in Top_*. Then, given any homology theory E_*, we have:

$$E_n(X,\, A) \cong E_n(X/A,\, *).$$

Proof. We exhibit the following excisive triad:

$$\left(C_j \quad ; \quad X_1 := \frac{X \coprod A \times [0,\, 2/3]}{\{x_0\} \times [0,\, 2/3]} \quad , \quad X_2 := \frac{A \times [1/3,\, 1]}{A \times \{1\} \cup \{x_0\} \times [0,\, 2/3]} \right).$$



One can notice that $X_1 \simeq X$ and $X_2 \simeq \{*\}$, and one can compute $X_1 \cap X_2 \cong A$. Exploiting then the excision axiom (H2) and the proposition 2.3.12, we get the claimed isomorphism:

$$E_n(X,\, A) \;\cong\; E_n(X_1,\, X_1 \cap X_2) \overset{(\text{H2})}{\cong} E_n(C_j,\, X_2) \;=\; E_n(C_j,\, *) \overset{3.36.}{=} E_n(X/A,\, *). \qquad \square$$

Theorem 3.1.8. A HOMOTOPY THEORY E_* ON $\mathsf{Top}_{\mathsf{rel}}$ DETERMINES AND IS CONVERSELY DETERMINED BY A REDUCED HOMOTOPY THEORY $\tilde{E}_*$ ON Top_*.

Proof. On the one hand, we consider a homotopy theory E_* on Top_{rel}. Let us show that it induces the reduced homotopy theory $\tilde{E}_*$ on Top_* with the relations $\tilde{E}_*(X,\, x_0) := E_*(X,\, x_0)$ and $\tilde{E}_*((X,\, x_0) \to (Y,\, y_0)) := E_*((X,\, x_0) \to (Y,\, y_0))$ for any pointed spaces $(X,\, x_0)$ and $(Y,\, y_0)$. In other words, we want to verity the axioms are satisfied. To lighten the notations, we will simply write $\tilde{E}_*(X)$ instead of $\tilde{E}_*(X,\, x_0)$ for any based space $(X,\, x_0)$ when the basepoint is clear.

($\tilde{\text{H}}$1) We will rather show ($\tilde{\text{H}}$1'). Consider a cofibration $j : (A,\, a_0) \rightarrowtail (X,\, x_0)$ in Top_*, and an integer n. Then, up to homeomorphism, we have $A \subseteq X$ and, one can apply the proposition 1.4.8 to the triad $(X,\, A,\, \{*\})$ in order to get the exact sequence:

$$E_n(A,\, *) \longrightarrow E_n(X,\, *) \longrightarrow E_n(X,\, A).$$

Hence, applying the lemma 3.1.7, it yields we well have the exact sequence of ($\tilde{\text{H}}$1'):

$$\tilde{E}_n(A) \longrightarrow \tilde{E}_n(X) \longrightarrow \tilde{E}_n(X/A).$$

($\tilde{\text{H}}$2) Let $n \in \mathbb{Z}$. As in the previous point, we use the proposition 1.4.8 but with the triad $(CX,\, X,\, \{*\})$, and it gives the exact sequence:

$$E_{n+1}(CX,\, *) \longrightarrow E_{n+1}(CX,\, X) \longrightarrow E_n(X,\, *) \longrightarrow E_n(CX,\, *)$$

We notice that $E_*(\{*\},\, *) = 0$ since the lemma 3.1.6 gives $E_*(\{*\}) = E_*(\{*\}) \oplus E_*(\{*\},\, *)$. Moreover, we saw that on the one hand CX is contractible, and on the other hand the suspension ΣX is homeomorphic to CX/X. That implies $\tilde{E}_*(CX) := E_*(CX,\, *) = E_*(\{*\},\, *) = 0$, and $E_{n+1}(CX,\, X) \overset{3.1.7}{=} E_{n+1}(CX/X,\, *) = E_{n+1}(\Sigma X,\, *) =: \tilde{E}_{n+1}(\Sigma X)$. In other words, we have the exact sequence:

$$0 \longrightarrow \tilde{E}_{n+1}(\Sigma X) \longrightarrow \tilde{E}_n(X) \longrightarrow 0,$$

which induces the desired homeomorphism $\tilde{E}_{n+1}(\Sigma X) \cong \tilde{E}_n(X)$.

($\tilde{\text{H}}$3) Let $\{(X_i,\, x_i)\}_{i \in I_0}$ be collection of based spaces in Top_*. We notice the cofibrations $\{x_i\} \rightarrowtail X_i$ for $i \in I_0$ imply we have with the proposition 2.3.5 the cofibration $j : \coprod_i \{x_i\} \rightarrowtail \coprod_i X_i$. Using the lemma 3.1.7, we have in particular $E_*(\coprod X_i,\, \coprod_i \{x_i\}) \cong E_*((\coprod_i X_i)/(\coprod_i \{x_i\}))$, and it yields the wanted isomorphism:

$$\tilde{E}_*\left(\bigvee_{i \in I_0} X_i\right) = E_*\left(\left(\coprod_{i \in I_0} X_i\right) / \left(\coprod_{i \in I_0} \{x_i\}\right)\right) \cong E_*\left(\coprod_{i \in I_0} X_i,\, \coprod_{i \in I_0} \{x_i\}\right) \overset{(\text{H3})}{\cong} \bigoplus_{i \in I_0} E_*(X_i,\, x_i) =: \bigoplus_{i \in I_0} \tilde{E}_*(X_i).$$

($\tilde{\text{H}}$4) It is a particular case of (H4).

($\tilde{\text{H}}$5) We already computed $H_n(S^0,\, A)$ in 1.4.27. We have with the lemma 3.1.6 and the dimension axiom (H5):

$$A \oplus A \cong H_0(S^0,\, A) = H_0(\{*\},\, A) \oplus H_0(S^0,\, *) = A \oplus \tilde{H}_0(S^0),$$

so $\tilde{H}_0(S^0) \cong A$, and similarly for $n \neq 0$ we get $\tilde{H}_n(S^0) \cong 0$.

On the other hand, we consider a reduced homotopy theory $\tilde{E}_*$ on Top_*. For any topological space X, we define its **fixe pointed space** X_+ as being the pointed space $(X \coprod \{*\},\, *)$, and we have the following functor:

$$\left\{ \begin{array}{ccc} \mathit{Top} & \to & \mathit{Top}_* \\ X & \mapsto & X_+ := (X \coprod \{*\},\, *) \\ f : X \to Y & \mapsto & f_+ : X_+ \to Y_+ \end{array} \right.$$

where $f_+ := f$ on X, and maps $*$ to $*$. As a fixe pointed space X_+ is in $Ob(\mathit{Top}_*)$, one can compute mapping cones in Top_*:

$$\begin{tikzcd} A_+ \arrow[r, hook, "i_+"] \arrow[d, hook] & X_+ \arrow[d] \\ C(A_+) \arrow[r] & C_{i_+} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

This is why we suggest to build a homotopy theory defined as $E_*(X,\, A) := \tilde{E}_*(C_{i_+})$ and $E_*(f : (X,\, A) \to (Y,\, B)) := \tilde{E}_*(\theta : C_{i_+} \to C_{j_+})$, where i and j are the inclusions $A \hookrightarrow X$ and $B \hookrightarrow Y$, the pairs $(X,\, A)$ and $(Y,\, B)$ are in Top_{rel}, and the continuous map θ is induced by the pushout as follows:

$$\begin{tikzcd} A_+ \arrow[r, hook, "i_+"] \arrow[d, hook] & X_+ \arrow[d] \arrow[rr, bend left, "f_+"] & & Y_+ \arrow[d] \\ C(A_+) \arrow[r] \arrow[dr, bend right, "C(f_+|_{A_+ \to B_+})"'] & C_{i_+} \arrow[ul, phantom, "\ulcorner", very near start] \arrow[rr, dashed, "\exists!\theta"] & & C_{j_+}. \\ & C(B_+) \arrow[urr] & & \end{tikzcd}$$

Let us show this definition verifies the axioms.

(H1) Take a pair $(X,\, A)$ in Top_{rel}. The remark 3.1.2 applied to $i_+ : A_+ \hookrightarrow X_+$ gives us the following exact sequence:

$$\dots \longrightarrow \tilde{E}_n(A_+) \longrightarrow \tilde{E}_n(X_+) \longrightarrow \tilde{E}_n(C_{i_+}) \longrightarrow \tilde{E}_{n-1}(A_+) \longrightarrow \dots$$

We notice that $\tilde{E}_n(A_+) = E_n(A,\, \emptyset)$ because we have the pushout:

$$\begin{tikzcd} \emptyset_+ \cong \{*\} \arrow[r, hook] \arrow[d, equal] & A_+ \arrow[d, equal] \\ C(\emptyset_+) = \{*\} \arrow[r] & A_+ \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

and similarly $\tilde{E}_n(X_+) = E_n(X,\, \emptyset)$. It yields we have the wanted exact sequence:

$$\dots \longrightarrow E_n(A,\, \emptyset) \longrightarrow E_n(X,\, \emptyset) \longrightarrow E_n(X,\, A) \longrightarrow E_{n-1}(A,\, \emptyset) \longrightarrow \dots$$

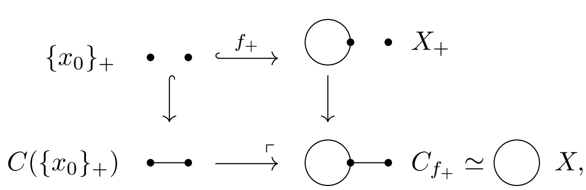
Notice that, secretly, the map $E_n(X,\, A) \to E_{n-1}(A,\, \emptyset)$ is well exactly $\partial_{n_{(X,\, A)}}$, where ∂_n is the natural transformation defined as follows for any relative space $(Y,\, B)$:

$$\begin{tikzcd} E_n(Y,\, B) = \tilde{E}_n(C_{j_+}) \arrow[r, "\tilde{E}_n(p)"] \arrow[dr, "\partial_{n_{(Y,\, B)}}"'] & \tilde{E}_n(\Sigma(B_+)) \arrow[d, "\cong\ (\mathrm{H2})"] \\ & E_{n-1}(B,\, \emptyset) = \tilde{E}_{n-1}(B_+) \end{tikzcd}$$

where $j_+ : B_+ \hookrightarrow Y_+$ and where $p : C_{j_+} \to \Sigma(B_+)$ is given by the cofiber sequence of j_+.

And the other axioms can be shown in a similar way as the ones of the reduced homology theory. □

Remark 3.1.9. Let us precise why the relation between theories given in the proof is well one-to-one. On the one hand, given a reduced homology theory $\tilde{E}_*$, we define a homology theory E_* with the relation $E_*(X,\, A) := \tilde{E}_*(C_{i_+})$ on Top_{rel}, and then we define a new reduced homology theory $\tilde{E}'_*$ with $\tilde{E}'_*(X,\, x_0) := E_*(X,\, x_0)$ on Top_*. We consider a pointed space $(X,\, x_0)$ and we notice that $\tilde{E}_*(X,\, x_0) = \tilde{E}_*(C_{f_+})$ where $f : \{x_0\} \hookrightarrow X$ because:



where $C(\{x_0\}_+) = C(\partial I) = I$ has been computed in the proof of 2.2.9. It implies we have:

$$\tilde{E}'_*(X,\, x_0) = E_*(X,\, x_0) = \tilde{E}_*(C_{f_+}) = \tilde{E}_*(X,\, x_0).$$

One can also show we have $\tilde{E}'_*(f) = \tilde{E}_*(f)$ for any continuous map f in Top_*, so it well yields $\tilde{E}'_* = \tilde{E}_*$.
On the other hand, considering a homology theory E_*, we define a reduced homology theory $\tilde{E}_*$ on Top_* as $\tilde{E}_*(X, x_0) = E_*(X, x_0)$, which induces a new homology theory E'_* such that $E'_*(X, A) := E_*(C_{f+})$ on Top_{rel}. For any topological space X, we have $E_*(X_+, *) = E_*(X \coprod\{*\}, \emptyset \coprod\{*\}) \cong E_*(X, \emptyset) \oplus E_*(\{*\}, *)$ with the additivity axiom (H3). We saw in the proof of ($\tilde{\text{H}}$2) that $E_*(\{*\}, *) = 0$, so it results $E_*(X_+, *) \cong E_*(X, \emptyset)$. Now we have:

$$E'_*(X, \emptyset) \;=\; \tilde{E}_*(X_+) \;=\; E_*(X_+, *) \;\cong\; E_*(X, \emptyset).$$

Then, for any pair (X, A) in Top_{rel}, applying the proposition 1.4.8 to the triad $(X, A, \emptyset)$, we get the exact sequence for any integer n:

$$\begin{tikzcd}
E_n(A) \arrow[r] \arrow[d, "\cong"] & E_n(X) \arrow[r] \arrow[d, "\cong"] & E_n(X, A) \arrow[r] \arrow[d] & E_{n-1}(A) \arrow[r] \arrow[d, "\cong"] & E_{n-1}(X) \arrow[d, "\cong"] \\
E'_n(A) \arrow[r] & E'_n(X) \arrow[r] & E'_n(X, A) \arrow[r] & E'_{n-1}(A) \arrow[r] & E'_{n-1}(X),
\end{tikzcd}$$

and we can conclude with the 5-lemma that $E_n(X, A) \cong E'_n(X, A)$. One can also show we have $\tilde{E}'_*(f) = \tilde{E}_*(f)$ for any continuous map f in Top_{rel}, so it well yields $E'_* = E_*$. Hence the one-to-one relation.

II. Singular homology via Dold-Thom

In this part, we begin to use what Albrecht Dold and René Thom really introduced in mid XX^{th} century. Find their work concerning the Dold-Thom theorem in the reference [DT58]. We will consider here an ordinary reduced homology theory $\widetilde{H}_*$ with the particular group of the integer $\mathbb{Z}$; we call **singular homology** such a homology. Before going further, given any pointed space X in Top_*, we admit that the functors $[\Sigma X, -]_*$ and $[\Sigma^2 X, -]_*$ from Top_* land respectively in Gr and Ab. Recall we saw in the proposition 1.2.12 they are homotopy invariant functors. In particular, note that $\pi_1(X) := [S^1, X]_* = [\Sigma S^0, X]_*$ is a group, and $\pi_n(X) = [\Sigma^2 S^{n-2}, X]_*$ is an abelian group for any integer $n \geq 2$.

1. Infinite symmetric product

We need a homotopically well-behaved functor: the infinite symmetric product $SP : \mathit{Top}_* \to \mathcal{A}$ from Top_* to a new category $\mathcal{A}$. This notion is deeply developed in [AGP02, 5.2]. First, let us define this new category $\mathcal{A}$: this is the category of the topological abelian monoids. More precisely, its objects are pointed spaces (A, a_0) in Top_* together with a multiplication $\mu_A : (A, a_0) \times (A, a_0) \to (A, a_0)$ in Top_* which is commutative, associative and unital (*i.e.* for any $a \in A$ we have $\mu_A(a, a_0) = a = \mu_A(a_0, a)$); and its morphisms are continuous pointed maps $f : A \to B$ for $A, B \in Ob(\mathcal{A})$, such that the following diagram commutes in Top_*:

$$\begin{tikzcd}
A \arrow[r, "f"] & B \\
A \times A \arrow[u, "\mu_A"] \arrow[r, "f\times f"'] & B \times B. \arrow[u, "\mu_B"']
\end{tikzcd} \tag{3.1}$$

In other words, this map f is a kind of group homomorphism. An example of topological abelian group is an abelian group endowed with the discret topology (where every subset is seen as open).
Now, let us define a functor $SP^n : \mathit{Top}_* \to \mathit{Top}_*$. Let (X, x_0) be a pointed space, and $n \geq 1$ an integer. We denote $\mathfrak{S}_n$ the symmetric group (endowed with the discrete topology) which acts on the n-times product $X^{\times n} := X \times \cdots \times X$ (together with the product topology) with:

$$\begin{cases} \mathfrak{S}_n \times X^{\times n} & \to \quad X^{\times n} \\ (\sigma, (x_1, ..., x_n)) & \mapsto \quad (x_{\sigma(1)}, ..., x_{\sigma(n)}). \end{cases}$$

We denote the orbit $SP^n(X, x_0) := X^{\times n}/\mathfrak{S}_n$ which is a topological space endowed with the quotient topology, and based on $[(x_0, ..., x_0)]$. In particular, the order of the coordinates in this space does not matter. We will simply write its elements under the form $[x_1, ..., x_n]$. Moreover, for any continuous map $f : (X, x_0) \to (Y, y_0)$ in Top_*, we define:

$$SP^n(f) : \begin{cases} SP^n(X, x_0) & \to \quad SP^n(Y, y_0) \\ [x_1, ..., x_n] & \mapsto \quad [f(x_1), ..., f(x_n)], \end{cases}$$

which is continuous and based, and one can easily verify that SP^n thereby defined is well a functor $\mathsf{Top}_* \to \mathsf{Top}_*$. Observe that, for $n = 1$, we get $SP^1 = id_{\mathsf{Top}_*}$ the identity functor, and for any $n \geq 1$ we have the embedding:

$$\begin{cases} SP^n(X,\, x_0) & \to & SP^{n+1}(X,\, x_0) \\ [x_1,\, ...,\, x_n] & \mapsto & [x_0,\, x_1,\, ...,\, x_n]. \end{cases}$$

Therefore, we can write $SP^n(X,\, x_0) \hookrightarrow SP^{n+1}(X,\, x_0)$, and we have the sequence:

$$(X,\, x_0) = SP^1(X,\, x_0) \hookrightarrow SP^2(X,\, x_0) \hookrightarrow SP^3(X,\, x_0) \hookrightarrow ... \,.$$

Now, let us define SP.

Definition 3.2.1. *The* ***infinite symmetric product*** *is the functor $SP : \mathsf{Top}_* \to \mathcal{A}$ defined for any pointed space X and continuous map $f : X \to Y$ in Top_* by:*

$$SP(X) := \bigcup_{n\geq 1} SP^n(X), \qquad \text{and} \qquad SP(f) : \begin{cases} \bigcup_n SP^n(X) & \to & \bigcup_n SP^n(Y) \\ [x_1,\, ...,\, x_n] & \mapsto & [f(x_1),\, ...,\, f(x_n)]. \end{cases}$$

where $SP(X)$ is endowed with the union topology (namely, a set C is closed if, and only if, the intersection $C \cap SP^n(X)$ is closed for any integer $n \geq 1$). Moreover, if we denote x_0 the basepoint of X, then the basepoint of $SP(X)$ is $[x_0] = [x_0,\, ...,\, x_0]$.

Remark 3.2.2. Indeed, it is clear that SP is well a functor, but let us verify that in addition it lands in $\mathcal{A}$. Take a pointed space X in Top_* and denote x_0 its basepoint. We can endow $SP(X)$ with the following multiplication:

$$\mu_X : \begin{cases} SP(X) \times SP(X) & \to & SP(X) \\ ([x_1,\, ...,\, x_n],\, [x'_1,\, ...,\, x'_m]) & \mapsto & [x_1,\, ...,\, x_n,\, x'_1,\, ...,\, x'_m], \end{cases}$$

which is commutative because, as we saw previously, the order does not matter; also clearly associative, and unital because $[x_0,\, ...,\, x_1] = [x_1]$ for any $x_1 \in X$. Furthermore, for any continuous map $f : X \to Y$ in Top_*, we have:

$$\forall [x],\, [x'] \in SP^1(X), \qquad \begin{aligned} \mu_Y \circ (f \times f)([x],\, [x']) &= \mu_Y([f(x)],\, [f(x')]) \\ &= [f(x),\, f(x')] \\ &= (f \times f)([x,\, x']) \\ &= (f \times f) \circ \mu_X([x],\, [x']), \end{aligned}$$

and similarly in greater dimensions. It means the diagram (3.1) commutes, and therefore SP is well a functor from Top_* to $\mathcal{A}$.

Remark 3.2.3. We saw the page 3 some examples of free functors, and we could add SP to this list.

Proposition 3.2.4. The functor $SP : \mathsf{Top}_* \to \mathcal{A}$ is homotopically well-behaved, in the sense that:

$$f \simeq_* g \in Mor(\mathsf{Top}_*) \qquad \Rightarrow \qquad SP(f) \simeq_* SP(g).$$

Proof. Let $f \simeq_* g : X \to Y$ in Top_*, and $H : X \times I \to Y$ the associated pointed homotopy. Then the map:

$$\begin{cases} SP(X) \times I & \to & SP(X \times I) & \to & SP(Y) \\ ([x_1,\, ...,\, x_n],\, t) & \mapsto & [(x_1,\, t),\, ...,\, (x_n,\, t)] & \mapsto & SP(H)([(x_1,\, t),\, ...,\, (x_n,\, t)]), \end{cases}$$

is a pointed homotopy that gives the wanted relation $SP(f) \simeq_* SP(g)$. □

One can show the following interesting result with the CW-complexes, for instance in [AGP02, 5.2.2].

Proposition 3.2.5. If X is a pointed CW-complex, then $SP(X)$ is so. □

The following proposition gives the relation $S^1 \simeq SP(S^1)$, but we have to pay attention since it does not stand in higher dimensions: $S^n \not\simeq SP(S^n)$ for $n > 1$. There, we consider S^1 as subspace of the complex plan $\mathbb{C}$, together with the usual multiplication $\cdot$ and the basepoint $1 \in \mathbb{C}$.

Proposition 3.2.6. THE INCLUSION $i : S^1 = SP^1(S^1) \hookrightarrow SP(S^1)$ IS A HOMOTOPY EQUIVALENCE.

Proof. We define the map:

$$m : \begin{cases} SP(S^1) & \to & S^1 \\ [x_1, ..., x_n] & \mapsto & \mu_{S^1}(x_1, \mu_{S^1}(x_2, ...)) = x_1 \cdot ... \cdot x_n, \end{cases}$$

which is in $\mathcal{A}$ because, μ_{S^1} being in *Top*$_*$, it is also in *Top*$_*$, and it satisfies the diagram (3.1). We observe that $m \circ i = id_{S^1}$. Let us show that $i \circ m \simeq_* id_{SP(S^1)}$. Remark that:

$$\forall [x_1, ..., x_n] \in SP(S^1), \qquad i \circ m([x_1, ..., x_n]) = i(x_1 \cdot ... \cdot x_n) = [x_1 \cdot ... \cdot x_n] = [x_1 \cdot ... \cdot x_n, 1, ..., 1].$$

That implies we have the wanted homotopy equivalence thanks to the following pointed homotopy:

$$\begin{cases} SP(S^1) \times I & \to & SP(S^1), \\ ([x_1, ..., x_n], t) & \mapsto & [x_1^t \cdot ... \cdot x_n^t, 1, ..., 1]. \end{cases}$$

□

2. Quasifibration

We defined the n-th relative homotopy group on *Top*$_{rel}$ in 2.2.7 (which is an extension of the one on *Top*$_*$ defined in 1.2.16). Now, with this notion, Dold and Thom gave the following definition of quasifibration. For any based space (X, x_0) in *Top*$_*$, when the basepoint is clear, we will just write $\pi_n(X)$ instead of $\pi_n(X, x_0)$ to lighten the notations.

Definition 3.2.7. Dold-Thom. *A continuous map $p : Y \to Z$ in Top is a* ***quasifibration*** *if for all $z_0 \in Z$ the induced map:*

$$p_* : \begin{cases} \pi_n(Y, p^{-1}(\{z_0\})) & \to & \pi_n(Z, z_0) \\ [f] & \mapsto & [p \circ f] \end{cases}$$

is an isomorphism for all integer $n \geq 0$.

Remark 3.2.8. In particular, if we apply the long exact sequence of homotopy groups seen in 2.2.9 to the pair $(Y, p^{-1}(\{z_0\}))$ in *Top*$_{rel}$, we have the exact sequence:

$$... \longrightarrow \pi_n(p^{-1}(\{z_0\})) \longrightarrow \pi_n(Y) \longrightarrow \pi_n(Y, p^{-1}(\{z_0\})) \overset{p_*}{\cong} \pi_n(Z, z_0) \longrightarrow \pi_{n-1}(p^{-1}(\{z_0\})) \longrightarrow ... \quad (3.2)$$

and we notice this is a sequence of pointed homotopy groups π_n : *Top*$_* \to$ *Set*$_*$.

Example. The following map is a quasifibration:

$$p : \begin{cases} \mathbb{R} & \to & S^1 \\ x & \mapsto & e^{2i\pi x}, \end{cases} \quad (3.3)$$

because $\pi_*(\mathbb{R}, p^{-1}(\{*\})) = \pi_*(\mathbb{R}, \mathbb{Z}) = \pi_*(\mathbb{R}/\mathbb{Z}) \cong \pi_*(S^1)$.

Now, let us state a proposition due to Dold and Thom concerning the quasifibrations. This result is shown in [AGP02, A.3.1].

Proposition 3.2.9. Dold-Thom. IN Top$_*$, CONSIDER A HAUSDORFF SPACE X AND A PATH-CONNECTED SUBSPACE $A \subseteq X$. IF THERE IS A COFIBRATION $A \rightarrowtail X$, THEN THE NATURAL PROJECTION $[-] : X \to X/A$ INDUCES A COFIBRATION $SP([-]) : SP(X) \to SP(X/A)$ IN Top (SECRETLY, WE FORGET THE BASEPOINT WITH A FORGETFUL FUNCTOR; SEE 3). IN PARTICULAR, WE HAVE FOR ANY INTEGER n:

$$\pi_n(SP(X), SP(A)) \cong \pi_n(SP(X/A)).$$

□

Corollary 3.2.10. LET X BE A POINTED SPACE IN Top$_*$. ASSUMING X HAUSDORFF AND PATH-CONNECTED, WE HAVE:

$$\pi_n(SP(X)) \cong \pi_{n+1}(SP(\Sigma X)).$$

Proof. As X and I are Hausdorff, the union $(X \times \{1\}) \cup (\{x_0\} \times I)$ and the product $X \times I$ are also Hausdorff, and it follows the quotient $CX := X \times I/((X \times \{1\}) \cup (\{x_0\} \times I))$ is so. Then, applying the previous proposition to the cofibration $X \rightarrowtail CX$, we have for all $n \in \mathbb{N}$:

$$\pi_n(SP(CX),\, SP(X)) \overset{3.2.9}{=} \pi_n(SP(CX/X)) \overset{1.4.17}{\cong} \pi_n(SP(\Sigma X)).$$

We know that CX is contractible (see 1.4.17), so, as SP is homotopically well-defined (cf. 3.2.4), we have $SP(CX) \simeq SP(\{*\})$. Now, for any integer $n \geq 1$, the set $SP^n(\{*\}) := \{*\}^{\times n}/S_n$ contains only one element, namely $[*, ..., *]$, which means $SP(\{*\})$ contains also only one element, that is its basepoint $[*]$, and it yields for any integer n:

$$\pi_n(SP(CX)) \;=\; \pi_n(SP(\{*\}) \;=\; \pi_n(\{*\}) \;=\; 0,$$

because there is only one map $S^n \to \{*\}$. Therefore, the exact sequence (3.2) gives for any integer n:

$$0 = \pi_{n+1}(SP(CX)) \longrightarrow \pi_{n+1}(SP(\Sigma X)) \longrightarrow \pi_n(SP(X)) \longrightarrow \pi_n(SP(CX)) = 0,$$

and we thus have the wanted isomorphism $\pi_{n+1}(SP(\Sigma X)) \cong \pi_n(SP(X))$ with the proposition 1.3.3. □

3. The Dold-Thom theorem

Now, let us take a look at the long-awaited Dold-Thom theorem.

Theorem 3.2.11. Dold-Thom. THE FUNCTORS $\tilde{H}_n(-;\, \mathbb{Z}) : \mathsf{Top}_* \to \mathsf{Ab}$ DEFINED FOR ALL $n \in \mathbb{Z}$ BY:

$$\forall X \in Ob(\mathsf{Top}_*), \qquad \tilde{H}_n(X;\, \mathbb{Z}) := \begin{cases} \pi_{n+1}(SP(\Sigma\tilde{X})) & \text{if } n \geq 0, \\ 0 & \text{if } n < 0, \end{cases}$$

FORM A ORDINARY REDUCED HOMOLOGY THEORY WITH GROUP $\mathbb{Z}$, WHERE $\tilde{X} \xrightarrow{\sim} X$ IS A CW-APPROXIMATION (SEE 2.1.16). IN PARTICULAR, IF X IS A PATH-CONNECTED CW-COMPLEX, WE HAVE FOR ANY INTEGER $n \geq 0$:

$$\tilde{H}_n(X;\, \mathbb{Z}) = \pi_n(SP(X)).$$

Proof. Let X bet a pointed space in Top_*. In order to lighten the redaction, we will write $\tilde{H}_n(X)$ instead of $\tilde{H}_n(X;\, \mathbb{Z})$, or sometimes even $\tilde{H}_*(X)$. One can see that $\tilde{H}_*(-)$ is a functor by composition of functors (see the remark 1.1.6), but let us show that it is homotopy invariant. If we take two pointed spaces $X \simeq_* X'$ in Top_*, then we have the weak equivalences $\tilde{X} \sim X \sim X' \sim \tilde{X}'$ with the CW-approximation and the proposition 1.2.28, which implies $\tilde{X} \simeq_* \tilde{X}'$ with the Whitehead theorem 2.1.15. It yields $SP(\Sigma\tilde{X}) \simeq_* SP(\Sigma\tilde{X}')$ because Σ and SP are homotopically well-behaved (see 1.4.22 and 3.2.4), and thus $\pi_*(SP(\Sigma\tilde{X})) = \pi_*(SP(\Sigma\tilde{X}'))$ because π_* is a homotopy invariant functor (cf. 1.2.18). Moreover, this functor is well-defined since, if we consider two different CW-approximations $\tilde{X}$ and $\hat{X}$ of X, we have the weak equivalences $\tilde{X} \sim X \sim \hat{X}$, and it results the homotopy equivalence $\tilde{X} \simeq \hat{X}$ by the Whitehead theorem 2.1.15. Now, let us verify the five axioms of a ordinary reduced homology theory.

($\tilde{\text{H}}$4) We consider a weak equivalence $f : X \to Y$ in Top_*. We have the following CW-approximation:

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\sim} & X \\ {\scriptstyle \tilde{f}}\downarrow & & \downarrow{\scriptstyle f}\,\wr \\ \tilde{Y} & \xrightarrow{\sim} & Y. \end{array}$$

which implies $\tilde{f}$ is also a weak equivalence, and it is even a homotopy equivalence due to the Whitehead theorem 2.1.15. Therefore, as the functors Σ and SP are homotopically well-behaved (see 1.4.22 and 3.2.4), and as π_* is a homotopy invariant functor (cf. 1.2.18), we well have the wanted isomorphism $\tilde{H}_n(f) := \pi_{n+1}(SP(\Sigma\tilde{f}))$ in Ab for all $n \geq 0$, and it is also obviously the case for any negative integer n.

(H̃1) Let $f : A \to X$ be a continuous map in Top_*. By CW-approximation and the theorem 2.3.16, we have the following commutative diagram:

$$\begin{tikzcd} A \arrow[rr, "f"] & & X \\ \tilde{A} \arrow[u, "\sim"] \arrow[rr, "\tilde{f}"] \arrow[dr, tail, "j"'] & & \tilde{X} \arrow[u, "\sim"'] \\ & M_{\tilde{f}}. \arrow[ur, "\simeq"'] & \end{tikzcd}$$

With the Barratt-Puppe sequence of the cofibration $j : \tilde{A} \rightarrowtail M_{\tilde{f}}$, we deduce the cofibration $\Sigma j : \Sigma\tilde{A} \rightarrowtail \Sigma M_{\tilde{f}}$ (see the remark 2.3.13). We want to apply the proposition 3.2.9 to this cofibration $\Sigma\tilde{A} \rightarrowtail \Sigma M_{\tilde{f}}$. Let us show that $\Sigma M_{\tilde{f}}$ is Hausdorff. We know thanks to the proposition 2.1.7 that the CW-complexes $\tilde{A}$ and $\tilde{X}$ are Hausdorff; and I is also Hausdorff as subset of $\mathbb{R}$ endowed with the usual topology. Now, in the pushout in Top_*:

$$\begin{tikzcd} \tilde{A} \arrow[r, "\tilde{f}"] \arrow[d, hook] & \tilde{X} \arrow[d] \\ \tilde{A} \times I \arrow[r] & M_{\tilde{f}}, \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

we have that $M_{\tilde{f}} := (\tilde{A} \times I) \vee_{\tilde{A}} \tilde{X}$ is Hausdorff for the property of being Hausdorff is preserved by product, coproduct and quotient. Then, for the same reasons, the suspension $\Sigma M_{\tilde{f}}$ is also Hausdorff. Moreover, the suspension $\Sigma\tilde{A}$ is path-connected as seen in 1.4.17. Thus, we can apply the proposition 3.2.9 and we have for any integer $n \geq 0$:

$$\begin{aligned} \pi_{n+1}\left(SP(\Sigma M_{\tilde{f}}),\, SP(\Sigma\tilde{A})\right) &\overset{3.2.9}{\cong} \pi_{n+1}\left(SP\left((\Sigma M_{\tilde{f}})/(\Sigma\tilde{A})\right)\right) \overset{2.3.12}{\cong} \pi_{n+1}\left(SP\left(C_{\Sigma j}\right)\right) \overset{(2.5)}{\cong} \pi_{n+1}\left(SP\left(\Sigma C_j\right)\right) \\ &\overset{2.1.16}{\cong} \pi_{n+1}\left(SP\left(\Sigma\widetilde{C_j}\right)\right) =: \tilde{H}_n(C_j) \overset{2.3.18}{=} \tilde{H}_n(C_{\tilde{f}}). \end{aligned}$$

One can see that we have the weak equivalence $C_{\tilde{f}} \sim C_f$, we then have by (H̃4) that $\tilde{H}_n(C_{\tilde{f}}) \cong \tilde{H}_n(C_f)$. Finally, we have due to (3.2) the wanted exact sequence for any integer $n \geq 0$:

$$\tilde{H}_n(X) := \pi_{n+1}(SP(\Sigma\tilde{X})) \longrightarrow \tilde{H}_n(Y) \overset{M_{\tilde{f}} \simeq \tilde{Y}}{=} \pi_{n+1}(SP(\Sigma M_{\tilde{f}})) \longrightarrow \tilde{H}_n(C_f) \cong \pi_{n+1}\left(SP(\Sigma M_{\tilde{f}}),\, SP(\Sigma\tilde{A})\right),$$

and we also have the result for $n < 0$ because the sequence $0 \to 0 \to 0$ is obviously exact.

(H̃2) Let X be a pointed space in Top_*. For all integer $n \geq 0$, knowing that $\Sigma\tilde{X}$ is Hausdorff and path-connected for the same reasons as in the previous point, the wanted isomorphism yields from the corollary 3.2.10:

$$\tilde{H}_n(X) \;:=\; \pi_{n+1}(SP(\Sigma\tilde{X})) \overset{3.2.10}{\cong} \pi_{n+2}(SP(\Sigma^2\tilde{X})) \overset{2.1.17}{=} \pi_{n+2}\left(SP\left(\Sigma(\widetilde{\Sigma X})\right)\right) \;=:\; \tilde{H}_{n+1}(\Sigma X).$$

For $n = -1$, as $\Sigma\tilde{X}$ is path-connected, we have that $SP(\Sigma\tilde{X})$ is also path-connected, so as π_0 is the set of the path components (see 1.2.21), it follows:

$$\tilde{H}_{-1}(X) := 0 = \pi_0(SP(\Sigma\tilde{X})) \overset{3.2.10}{\cong} \pi_1(SP(\Sigma^2\tilde{X})) \overset{2.1.17}{=} \pi_1\left(SP\left(\Sigma(\widetilde{\Sigma X})\right)\right) =: \tilde{H}_0(\Sigma X).$$

And, naturally, the result still stands for $n < -1$: $\quad H_n(X) := 0 =: H_{n+1}(\Sigma X)$.

(H̃3) We will show the result in particular case of 2 elements, but it can be generalized with transfinite induction arguments. Let X and Y be two pointed spaces in Top_*. We consider the natural projections $q : X \vee Y \to Y = (X \vee Y)/X$ and $r : X \vee Y \to (X \vee Y)/Y$, and we apply the exactness axiom (H̃1') to the cofibration $j : X \rightarrowtail X \vee Y$ to get the exact sequence:

$$\begin{tikzcd} \tilde{H}_*(X) \arrow[r, "\tilde{H}_*(j)"] & \tilde{H}_*(X \vee Y) \arrow[r, "\tilde{H}_*(q)"] \arrow[l, dashed, bend left, "\tilde{H}_*(r)"] & \tilde{H}_*((X \vee Y)/X) \;=\; \tilde{H}_*(Y). \end{tikzcd}$$

where $\tilde{H}_*(r)$ is a retraction. One can notice that $\tilde{H}_*(j)$ is injective because we have the following commutative diagram:

$$\begin{tikzcd} \tilde{H}_*(X) \arrow[r, "\tilde{H}_*(j)"] \arrow[dr, equal, bend right, "\tilde{H}_*(id)=id"'] & \tilde{H}_*(Y) \arrow[d, "\tilde{H}_*(r)"] \\ & \tilde{H}_*(X), \end{tikzcd}$$

and similarly $\tilde{H}_*(q)$ is surjective; so we can extend the exact sequence:

$$\begin{tikzcd} 0 \arrow[r] & \tilde{H}_*(X) \arrow[r, "\tilde{H}_*(j)"] & \tilde{H}_*(X \vee Y) \arrow[r, "\tilde{H}_*(q)"] \arrow[l, dashed, bend left, "\tilde{H}_*(r)"] & \tilde{H}_*(Y) \arrow[r] & 0. \end{tikzcd}$$

Now, applying the splitting lemma 1.3.12, we get the wanted isomorphism:

$$\tilde{H}_*(X \vee Y) \cong \tilde{H}_*(X) \oplus \tilde{H}_*(Y).$$

(H̃5) We well have $\tilde{H}_n(S^0) := 0$ for any $n < 0$. Now, for any integer $n \geq 0$, as S^0 is a CW-complex, we can consider the CW-approximation $\tilde{S}^0 = S^0$, and it induces:

$$\tilde{H}_n(S^0) := \pi_{n+1}(SP(\Sigma\tilde{S}^0)) = \pi_{n+1}(SP(\Sigma S^0)) \overset{1.4.19}{=} \pi_{n+1}(SP(S^1)) \overset{3.2.6}{=} \pi_{n+1}(S^1) = \begin{cases} \pi_1(S^1) \cong \mathbb{Z} & \text{if } n = 0, \\ 0 & \text{if } n > 0, \end{cases}$$

Indeed, to be brief, up to homotopy, to each loop in S^1 we can associate an integer in $\mathbb{Z}$: the algebraic number of "tours" around the center of S^1; and conversely any number k in $\mathbb{Z}$ corresponds to a unique class of loops: the one that contains a loop doing k "tours" around the center of S^1; and this relation is moreover one to one. Furthermore, we have $\pi_n(S^1) = 0$ for any $n > 1$ since the exact sequence (3.2) applied to the quasifibration $p : x \mapsto e^{ix}$ (cf. the example (3.3)) gives the exact sequence:

$$0 = \pi_n(\{*\}) = \pi_n(\mathbb{R}) \longrightarrow \pi_n(S^1)) \longrightarrow \pi_{n-1}(\mathbb{Z}) = 0,$$

where $\pi_{n-1}(\mathbb{Z}) = 0$ because any continuous map into $\mathbb{Z}$ is necessary constant to the given basepoint.

Note that, due to (H̃2), the functor $\tilde{H}_*(-;\mathbb{Z})$ lands in the category *Ab* of abelian groups just applying the fact that π_n lands in *Ab* for $n \geq 2$ (see page 47) and seeing besides that 0 is clearly an abelian group. Moreover, the particular case is an application of the corollary 3.2.10, noticing that we can take $\tilde{X} = X$ as CW-approximation of X when X is a CW-complex, and that CW-complexes are Hausdorff spaces (see 2.1.7). ☐

4. Examples of computation

To conclude this project, it might be interesting to give examples of computation of $\tilde{H}_*(X)$ for some X, namely the k-sphere S^k ($k \in \mathbb{N}$) and the torus $S^1 \times S^1$, where $\tilde{H}_*$ is any ordinary reduced homology theory with an abelian group A. Let n be any integer.

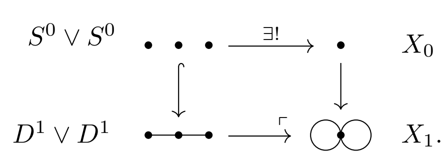
(I) Actually, we have almost already computed the first case: we know that $\tilde{H}_*$ is associated to one precise ordinary homology theory H_* due to the theorem 3.1.8, and it yields:

$$\tilde{H}_n(S^k) \oplus \begin{cases} A & \text{if } n = 0, \\ 0 & \text{otherwise,} \end{cases} \overset{(\text{H5})}{\cong} \tilde{H}_n(S^k) \oplus H_n(*) \overset{3.1.6}{=} H_n(S^k) \overset{1.4.27}{\cong} \begin{cases} A \oplus A & \text{if } k = n = 0, \\ A & \text{if } k = n > 0 \text{ or } k > n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, we have the first wanted result:

$$\tilde{H}_n(S^k) \cong \begin{cases} A & \text{if } k = n \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{3.4}$$

(II) Now, let us compute $\tilde{H}_*(T)$ where $T := S^1 \times S^1$ is the torus. We will use the fact that T is a pointed CW-complex with one 0-cell, two 1-cells and one 2-cell. Indeed, we denote $X_0 := (\{*\}, *)$ its 0-skeleton, the next one X_1 is given by the pushout in *Top*$_*$:



On the one hand, one can observe that $\tilde{H}_n(X_0) = \tilde{H}_n(\{*\}, *) = 0$ since we have $\tilde{H}_n(S^0) = \tilde{H}_n(S^0 \vee (\{*\}, *)) = \tilde{H}_n(S^0) \oplus \tilde{H}_n(\{*\}, *)$ due to (H3). On the other hand, we saw in the proof of the theorem 2.2.9 that the reduced cone of $S^0 = \partial I$ is $C(S^0) = I = D^1$. One can similarly compute the reduced cone of $S^0 \vee S^0$ and then get $C(S^0 \vee S^0) = D^1 \vee D^1$. It implies by unicity of the pushout up to homeomorphism (see 1.1.26) that the 1-skeleton X_1 is no more than the mapping cone C_f if we denote by f the map $S^0 \vee S^0 \to X_0$:

$$\begin{tikzcd} S^0 \vee S^0 \arrow[r, "f"] \arrow[d] & X_0 \arrow[d] \\ D^1 \vee D^1 = C(S^0 \vee S^0) \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & X_1 \cong C_f. \end{tikzcd}$$

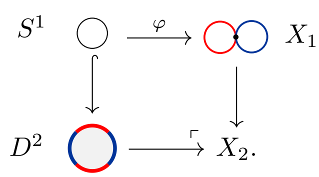
Consequently, with the remark 3.1.2 applied to f, we obtain the following exact sequence:

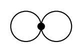
$$0 = \tilde{H}_n(X_0) \longrightarrow \tilde{H}_n(C_f) \cong \tilde{H}_n(X_1) \longrightarrow \tilde{H}_{n-1}(S^0 \vee S^0) \overset{(\tilde{\text{H}}3)}{\cong} \tilde{H}_{n-1}(S^0) \oplus \tilde{H}_{n-1}(S^0) \longrightarrow \tilde{H}_{n-1}(X_0) = 0.$$

Hence, we deduce the wanted result from the previous computation:

$$\tilde{H}_n(X_1) \overset{1.3.3}{\cong} \tilde{H}_{n-1}(S^0) \oplus \tilde{H}_{n-1}(S^0) \overset{(3.4)}{=} \begin{cases} A \oplus A & \text{if } n = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{3.5}$$

Next, let us compute the 2-skeleton X_2 using one 2-cell:



Here, the image of the attaching map φ "travels" in  "making twice the 8": first the left circle, then the right one, and everything again one more time but crossing the two circles in the reversed rotation. So, computing the pushout X_2, we attach in D^2 respectively the two red parts and the two blue parts together with the red and the blue circles. It follows obtain $X_2 = T$ because we recognize the standard identification that defines the torus. Now, notice that the disk D^2 is homeomorphic to the reduced cone $C(S^1)$, so that we have again $C_\varphi = X_2$, as it was for X_1. Therefore, on the one hand, due to the remark 3.1.2 applied to φ, we obtain the following exact sequence for $n \neq 1$ and $n \neq 2$:

$$0 \overset{(3.5)}{=} \tilde{H}_n(X_1) \longrightarrow \tilde{H}_n(C_\varphi) \cong \tilde{H}_n(X_2) \longrightarrow \tilde{H}_{n-1}(S^1) \overset{(3.4)}{=} 0,$$

which implies $\tilde{H}_n(X_2) = 0$ for any integer $n \neq 1$ and $n \neq 2$. On the other hand, we have due to the same remark the following exact sequence:

$$\tilde{H}_2(X_1) \longrightarrow \tilde{H}_2(C_\varphi) \longrightarrow \tilde{H}_1(S^1) \longrightarrow \tilde{H}_1(X_1) \longrightarrow \tilde{H}_1(C_\varphi) \longrightarrow \tilde{H}_0(S^1),$$

which can be simplified with (3.4) and (3.5) into the exact sequence:

$$0 \longrightarrow \tilde{H}_2(X_2) \longrightarrow A \longrightarrow A \oplus A \longrightarrow \tilde{H}_1(X_2) \longrightarrow 0.$$

A similar reasoning as in the third point (III) of the proof of 1.4.27 leads us to the results $\tilde{H}_2(X_2) \cong A$ and $\tilde{H}_1(X_2) \cong A \oplus A$. Hence, we have just computed the homology of the torus:

$$\tilde{H}_n(T) = \tilde{H}_n(X_2) \cong \begin{cases} A \oplus A & \text{if } n = 1, \\ A & \text{if } n = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Appendices

A- Categories used in this project

Here is a list of all the categories used in this project:

NOTATION	OBJECTS	MORPHISMS	ISOMORPHISMS
Set	Sets	Maps of sets	Bijections
Set_*	Pointed sets	Pointed maps of sets	Pointed bijections
Top	Topological spaces	Continuous maps	Homeomorphisms
Top_*	Pointed spaces	Continuous pointed maps	Pointed homeomorphisms
Top_{rel}	Relative spaces	Continuous relative maps	Relative homeomorphisms
CW	CW-complexes	Continuous maps between CW-c.	Homeo. between CW-c.
CW_*	Pointed CW-complexes	Pointed morphisms of CW	Pointed isomorphisms of CW
Gr	Groups	Group homomorphisms	Group isomorphisms
Ab	Abelian groups	Abelian group homomorphisms	Abelian group isomorphisms

Table 1 — Categories used in this project

B- Proof that π_n is a group

Proposition 1.2.19. Let be $n \geq 1$ a positive integer, and $(X,\, x_0)$ a pointed topological space. The n-th homotopy group $\pi_n(X,\, x_0)$ is a group.

Proof. We will follow the steps given at the page 12.

(i) We want to show that $G := [(I^n,\, \partial I^n),\, (X,\, x_0)]$ is a group. For any two continuous maps $f,\, g : (I^n,\, \partial I^n) \to (X,\, x_0))$ in Top_{rel}, we define the operation $+$ in $\mathsf{Top}_{rel}((I^n,\, \partial I^n),\, (X,\, x_0))$ as follows:

$$\forall (s_1, \ldots, s_n) \in I^n, \qquad (f+g)(s_1,\, s_2, \ldots, s_n) := \begin{cases} f(2s_1,\, s_2, \ldots, s_n) & \text{for } s_1 \in [0,\, 1/2], \\ g(2s_1 - 1,\, s_2, \ldots, s_n) & \text{for } s_1 \in [1/2,\, 1], \end{cases}$$

where for $s_1 = 1/2$ we well have $f(2s_1,\, s_2, \ldots, s_n) = x_0 = g(2s_1 - 1,\, s_2, \ldots, s_n)$. We notice that this operation $+$ is well closed: $f + g \in \mathsf{Top}_{rel}((I^n,\, \partial I^n),\, (X,\, x_0))$. Now, by quotient, we want to define a similar operation $\cdot$ on G as follows:

$$[f] \cdot [g] := [f + g],$$

but we need to verify that such an operation is well defined. First, we can see that this operation is closed: as $f + g \in \mathsf{Top}_{rel}((I^n,\, \partial I^n),\, (X,\, x_0))$, we well have $[f + g] \in G$. Now, if we take $f' \in [f]$ and $g' \in [g]$, we have $f' \simeq_{\partial I^n} f$ and $g' \simeq_{\partial I^n} g$, and we note H_f and H_g the corresponding relative homotopies. We notice that the relative homotopy H defined by:

$$\forall t \in I, \qquad H(-,\, t) := H_f(-,\, t) + H_g(-,\, t),$$

(where the operation $+$ is defined above) gives us the relation $f' + g' \simeq_{\partial I^n} f + g$. This means that the operation $\cdot$ is well defined:

$$[f'] \cdot [g'] = [f' + g'] = [f + g] = [f] \cdot [g].$$

Next, let us show that the operation $\cdot$ is a group law. Let us consider three continuous maps $f, g, h \in \mathit{Top}_{rel}((I^n, \partial I^n), (X, x_0))$ and a point $(s_1, s_2, \dots, s_n) \in I^n$. On the one hand, we have:

$$
\begin{aligned}
((f+g)+h)(s_1, s_2, \dots, s_n) &= \begin{cases} (f+g)(2s_1, s_2, \dots, s_n) & \text{for } s_1 \in [0, 1/2], \\ h(2s_1 - 1, s_2, \dots, s_n) & \text{for } s_1 \in [1/2, 1], \end{cases} \\
&= \begin{cases} f(4s_1, s_2, \dots, s_n) & \text{for } s_1 \in [0, 1/4], \\ g(4s_1 - 1, s_2, \dots, s_n) & \text{for } s_1 \in [1/4, 1/2], \\ h(2s_1 - 1, s_2, \dots, s_n) & \text{for } s_1 \in [1/2, 1], \end{cases}
\end{aligned}
$$

and on the other hand:

$$
\begin{aligned}
(f+(g+h))(s_1, s_2, \dots, s_n) &= \begin{cases} f(2s_1, s_2, \dots, s_n) & \text{for } s_1 \in [0, 1/2], \\ (g+h)(2s_1 - 1, s_2, \dots, s_n) & \text{for } s_1 \in [1/2, 1], \end{cases} \\
&= \begin{cases} f(2s_1, s_2, \dots, s_n) & \text{for } s_1 \in [0, 1/2], \\ g(4s_1 - 2, s_2, \dots, s_n) & \text{for } s_1 \in [1/2, 3/4], \\ h(4s_1 - 3, s_2, \dots, s_n) & \text{for } s_1 \in [3/4, 1]. \end{cases}
\end{aligned}
$$

So the following relative homotopy defined by: $\forall (s_1, \dots, s_n) \in I^n$, $\forall t \in I$,

$$
H((s_1, s_2, \dots, s_n), t) := \begin{cases} f\left(\frac{4}{t+1} s_1, s_2, \dots, s_n\right) & \text{for } s_1 \in \left[0, \frac{t+1}{4}\right], \\ g(4s_1 - 1 - t, s_2, \dots, s_n) & \text{for } s_1 \in \left[\frac{t+1}{4}, \frac{t+2}{4}\right], \\ h\left(\frac{4}{-t+2} s_1 + \frac{t+2}{t-2}, s_2, \dots, s_n\right) & \text{for } s_1 \in \left[\frac{t+2}{4}, 1\right]. \end{cases}
$$

gives us the relation $(f+g)+h \simeq_{\partial I^n} f+(g+h)$. This implies that we have:

$$
([f] \cdot [g]) \cdot [h] = [(f+g)+h] = [f+(g+h)] = [f] \cdot ([g] \cdot [h]),
$$

which means that the operation $\cdot$ is associative. Furthermore, we can see that $[e : (s_1, \dots, s_n) \mapsto x_0]$ the identity element: indeed, we have the relation $f + e \simeq_{\partial I^n} f$ with the relative homotopy defined by:

$$
\forall (s_1, \dots, s_n) \in I^n, \forall t \in I, \qquad H'((s_1, s_2, \dots, s_n), t) := \begin{cases} f\left(\frac{2}{t+1} s_1, s_2, \dots, s_n\right) & \text{for } s_1 \in \left[0, \frac{t+1}{2}\right], \\ x_0 & \text{for } s_1 \in \left[\frac{t+1}{2}, 1\right], \end{cases}
$$

which means that we have $[f] \cdot [e] = [f+e] = [f]$, and in a same way $[e] \cdot [f] = [f]$. Finally, let us show that $[\tilde{f} : (s_1, s_2, \dots, s_n) \mapsto f(1 - s_1, s_2, \dots, s_n)]$ is the inverse element of $[f]$. We have:

$$
(\tilde{f} + f)(s_1, s_2, \dots, s_n) := \begin{cases} f(-2s_1 + 1, s_2, \dots, s_n) & \text{for } s_1 \in [0, 1/2], \\ f(2s_1 - 1, s_2, \dots, s_n) & \text{for } s_1 \in [1/2, 1], \end{cases}
$$

and we can consider the relative homotopy defined as follows: $\forall (s_1, \dots, s_n) \in I^n$, $\forall t \in I$,

$$
H''((s_1, s_2, \dots, s_n), t) := \begin{cases} f(-2s_1 + 1, s_2, \dots, s_n) & \text{for } s_1 \in \left[0, \frac{1-t}{2}\right], \\ f(2s_1 + 2t - 1, s_2, \dots, s_n) & \text{for } s_1 \in \left[\frac{1-t}{2}, 1 - t\right], \\ x_0 & \text{for } s_1 \in [1 - t, 1], \end{cases}
$$

which gives us the relation $\tilde{f} + f \simeq_{\partial I^n} e$. We deduce that $[\tilde{f}] \cdot [f] = [\tilde{f} + f] = [e]$, and in a same way $[f] \cdot [\tilde{f}] = [e]$. Therefore, $[\tilde{f}]$ is the inverse element of $[f]$, and we have shown that $\cdot$ is a group law. Hence $(G, \cdot)$ is a group.

(II) There is an obvious homeomorphism between $I^n - (\partial I^n)$ and the n-dimentional hyperspace $\mathbb{R}^n$ using for instance the continuous map $\arctan(-)/\pi + 1/2$ on each coordonate. There is also a homeomorphism between $\mathbb{R}^n$ and $S^n - \{*\}$ considering for example the stereographic projection. So, by composition we have a homeomorphism between $I^n - (\partial I^n)$ and $S^n - \{*\}$; and by continuity, we can extend the homeomorphism to I^n, mapping the boundary ∂I^n to the point $*$. Then, we can deduce the following homeomorphisms in Top:

$$
I^n / \partial I^n \cong (I^n - (\partial I^n)) \cup (\partial I^n / \partial I^n) = (I^n - (\partial I^n)) \cup \{[0]\} \cong S^n,
$$

and finally we conclude with the wanted homeomorphism in Top_{rel}:

$$
(I^n / \partial I^n, [0]) \cong (S^n, *).
$$

(III) We apply the remark 1.1.9 with the contravariant functor $[-, (X, x_0)]$ to the bijection provided by (II), and we obtain the bijection in *Set*:

$$\pi_n(X, x_0) := [(S^n, *), (X, x_0)] \cong [(I^n/\partial I^n, [0]), (X, x_0)]. \tag{6}$$

With a similar reasoning as in the proof of the proposition 1.2.15, we get the bijection:

$$[(I^n, \partial I^n), (X, x_0)] \cong [(I^n/\partial I^n, [0]), (X, x_0)]. \tag{7}$$

Finally, we gather the informations provided by (6) and (7) and we obtain the bijection:

$$\pi_n(X, x_0) \cong [(I^n, \partial I^n), (X, x_0)], \tag{8}$$

where the second set is the group G as seen in the first point (I). That is why $\pi_n(X, x_0)$ is also a group together with the operation induced by the bijection (8). $\square$

Index

References

[AGP02] Marcelo Aguilar, Samuel Gitler, and Carlos Prieto. *Algebraic Topology from a Homotopical Viewpoint.* Universitext. Springer-Verlag, New-York, 2002. Translated from the Spanish by Stephen Bruce Sontz.

[Bor94] Francis Borceaux. *Handbook of Categorical Algebra 1*, volume 50 of *Encyclopedia of mathematics and its applications.* Cambridge University Press, 1994.

[DT58] Albrecht Dold and Rene Thom. *Quasifaserungen und Unendliche Symmetrische Produkte*, volume 67. Annals of Mathematics, second edition, 1958. Notes.

[ES45] Samuel Eilenberg and Norman E. Steenrod. *Axiomatic Approach to Homology Theory*, volume 31. Proc. Nat. Acad. Sci. U.S.A., 1945.

[ES52] Samuel Eilenberg and Norman E. Steenrod. *Foundations of Algebraic Topology.* Princeton University Press, 1952.

[Han] Soren Hansen. *CW-complexes.* Notes.

[Hat01] Allen Hatcher. *Algebraic Topology.* Cambridge University Press, 2001.

[May99] Jon P. May. *A Concise Course in Algebraic Topology*, 1999.

[Pic92] Renzo A. Piccinini. *Lectures on homotopy theory*, volume 171 of *North-Holland Mathematics Studies.* North-Holland Publishing Co., Amsterdam, 1992.

[Rot09] Joseph J. Rotman. *An Introduction to Homological Algebra.* Universitext. Springer Science+Business Media, New-York, second edition, 2009.

[Sim07] J. Simon. *Urysohn's Lemma*, 5th edition, 2007.

[Tod63] Hirosi Toda. *Note on Cohomology Ring of Certain Spaces.* Proc. A.M.S. 14, 1963.