NonLocal Boxes
& Communication Complexity

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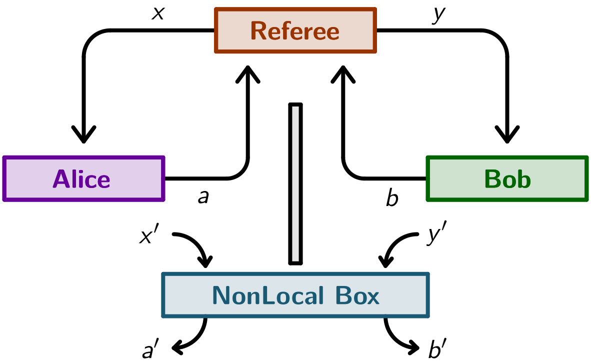
— *Part* 1 —

Definitions & Notations

CHSH Game

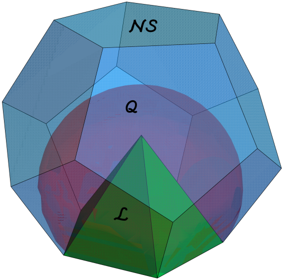


Win at CHSH. $a \oplus b = x\,y$.

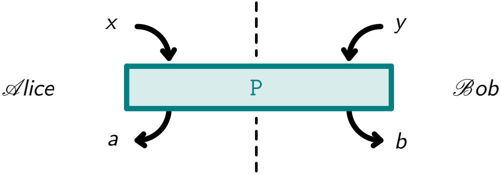
Win at CHSH'. $a \oplus b = (x \oplus 1)\,(y \oplus 1)$.

Strategies

- **Deterministic strategies.** $\rightsquigarrow \max \mathbb{P}(\text{win}) = 75\%$.
- **Classical strategies $\mathcal{L}$.** $\rightsquigarrow \max \mathbb{P}(\text{win}) = 75\%$.
- **Quantum strategies $\mathcal{Q}$.** $\rightsquigarrow \max \mathbb{P}(\text{win}) = \cos^2\left(\frac{\pi}{8}\right) \approx 85\%$.
- **Non-signalling strategies $\mathcal{NS}$.** $\rightsquigarrow \max \mathbb{P}(\text{win}) = 100\%$.



NonLocal Box

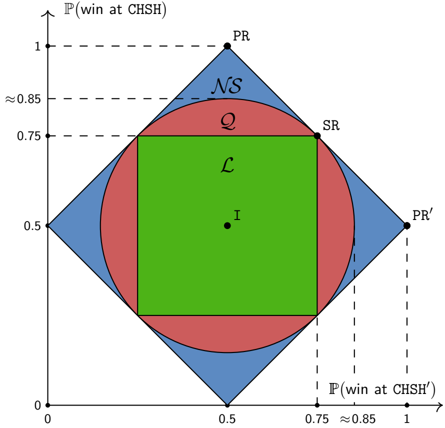


Definition. *A* **nonlocal box** *is a function:*

$$\mathtt{P} : \begin{cases} \{0,1\}^4 & \longrightarrow & [0,1] \\ (a,b,x,y) & \longmapsto & \mathtt{P}\big(a,b \,\big|\, x,y\big). \end{cases}$$

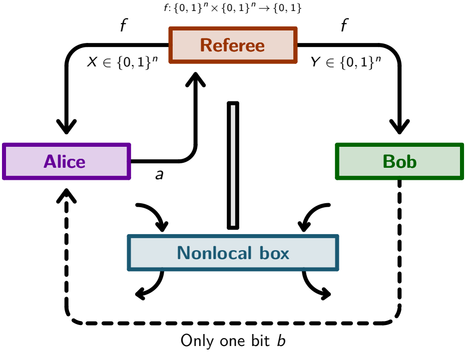
such that (i) $\mathtt{P}$ *is a conditional probability distribution and (ii)* $\mathtt{P} \in \mathcal{NS} \backslash \mathcal{L}$.

Examples



- $\mathtt{PR}\big(a, b \,|\, x, y\big) := \begin{cases} \frac{1}{2} & \text{si } a \oplus b = x\,y, \\ 0 & \text{otherwise.} \end{cases}$
- $\mathtt{PR}'\big(a, b \,|\, x, y\big) := \begin{cases} \frac{1}{2} & \text{si } a \oplus b = (x \oplus 1)\,(y \oplus 1), \\ 0 & \text{otherwise.} \end{cases}$
- $\mathtt{SR}\big(a, b \,|\, x, y\big) := \begin{cases} \frac{1}{2} & \text{si } a = b, \\ 0 & \text{otherwise.} \end{cases}$
- $\mathtt{I}\big(a, b \,|\, x, y\big) := \frac{1}{4}$

Communication Complexity



Win $\iff a = f(X, Y)$.

Def. *A function f is said to be* **trivial** *(in the sense of communication complexity) if Alice knows any value $f(X, Y)$ with only one bit transmitted between Alice and Bob.*

Ex. For $n = 2$, $X = (x_1, x_2)$, $Y = (y_1, y_2)$:

- $f := x_1 \oplus y_1 \oplus x_2 \oplus y_2 \oplus 1$ is trivial.
- $g := (x_1\, x_2) \oplus (y_1\, y_2)$ is trivial.
- $h := (x_1\, y_1) \oplus (x_2\, y_2)$ is NOT trivial.

Def. *A box* $\texttt{P}$ *is said to be* **collapsing** *(or* trivial*) if using copies of this box* $\texttt{P}$ *any Boolean function f is trivial, with probability $\geq q > \frac{1}{2}$.*

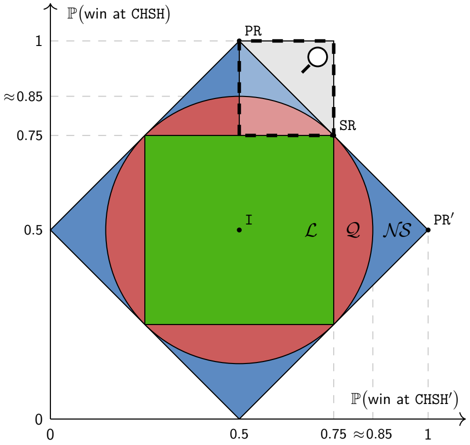
Ex. Link with previous boxes:

- The boxes $\texttt{PR}$ and $\texttt{PR}'$ are collapsing.
- The boxes $\texttt{SR}$ and $\texttt{I}$ are NOT collapsing.

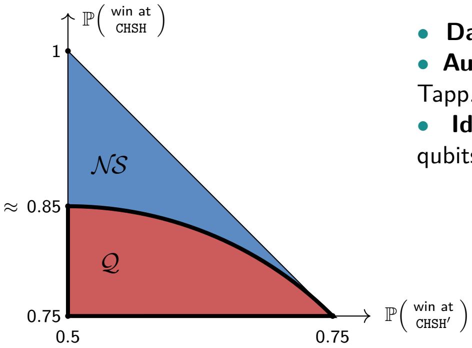
— *Part 2* —

Historical Overview

Goal. Show that quantum boxes are **non-collapsing** but that post-quantum boxes are **collapsing**.

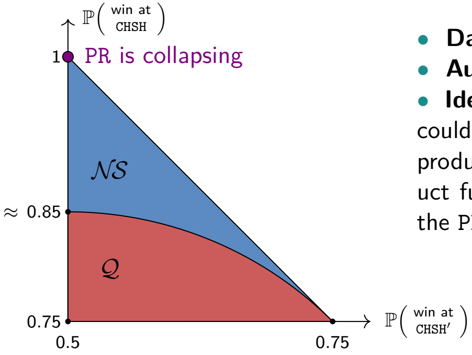


1999: Quantum boxes are non-collapsing



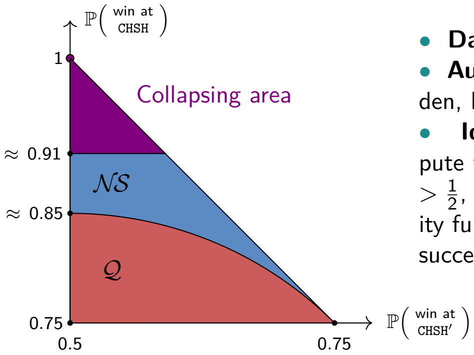
- **Date.** 1999 [1].
- **Authors.** Cleve, van Dam, Nielson, Tapp.
- **Ideas.** (1) Prove the result with qubits, (2) Go back to bits.

1999: The PR box is collapsing



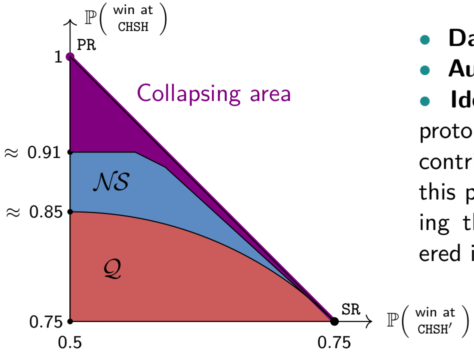
- **Date.** 1999 [2].
- **Author.** van Dam.
- **Ideas.** (1) Any Boolean function f could be written in terms of an inner product function, (2) Any inner product function is trivial using copies of the PR box.

2006: Boxes above $\approx$ 91% are collapsing



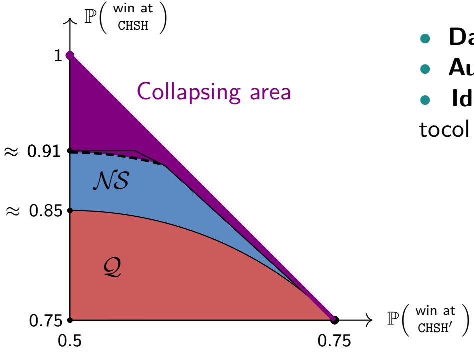
- **Date.** 2006 [3].
- **Authors.** Brassard, Buhrman, Linden, Méthot, Tapp, Unger.
- **Ideas.** (1) Distributively compute the given function f with proba $> \frac{1}{2}$, (2) Inductively apply the majority function `Maj` in order to boost the success probability.

2009: Correlated boxes are collapsing



- **Date.** 2009 [4].
- **Authors.** Brunner, Skrzypczyk.
- **Ideas.** (1) Introduce a distillation protocol, cf. generalization in "Our contribution", (2) Inductively apply this protocol many times until reaching the "collapsing triangle" discovered in 2006.

2018: Boxes above an ellipse are collapsing

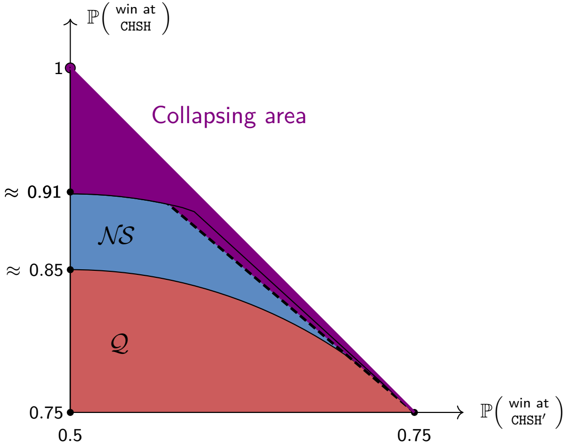


- **Date.** 2018 [5].
- **Author.** Broadbent, Proulx.
- **Idea.** Generalize BBLMTU's protocol (cf. 2006).

— *Part 3* —

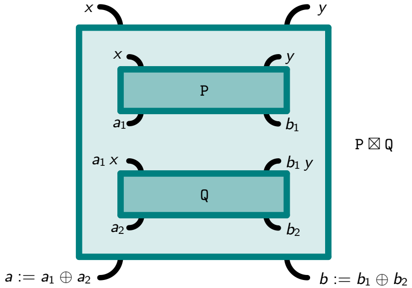
Our Contribution: Algebra of Boxes

Our contribution



Algebra of boxes

Recall. *A* nonlocal box $\mathtt{P}$ *is a conditional probability distribution* $(a,b,x,y) \in \{0,1\}^4 \mapsto \mathtt{P}(a,b \,|\, x,y) \in [0,1]$ *such that* $\mathtt{P} \in \mathcal{NS} \backslash \mathcal{L}$.



$$\mathtt{P} \boxtimes \mathtt{Q}\Big(a,\ b \,\Big|\, x,\ y\Big) := \sum_{a_1, b_1 \in \{0,1\}} \mathtt{P}\Big(a_1,\ b_1 \,\Big|\, x,\ y\Big) \times \mathtt{Q}\Big(a \oplus a_1,\ b \oplus b_1 \,\Big|\, a_1 x,\ b_1 y\Big)$$

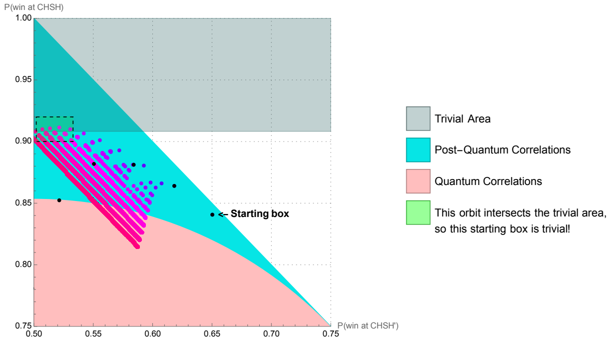
Algebra of boxes. The vector space $\mathcal{B} := \mathcal{F}\big(\{0,1\}^4, \mathbb{R}\big)$ endowed with the operations $\{+, \cdot, \boxtimes\}$ defines a non-commutative and non-associative algebra.

Orbit of a box

Orbit of order k**.** $\texttt{Orbit}_k(\mathtt{P}) := \Big\{\text{products of exactly } k \text{ times the term } \mathtt{P}\Big\}$.

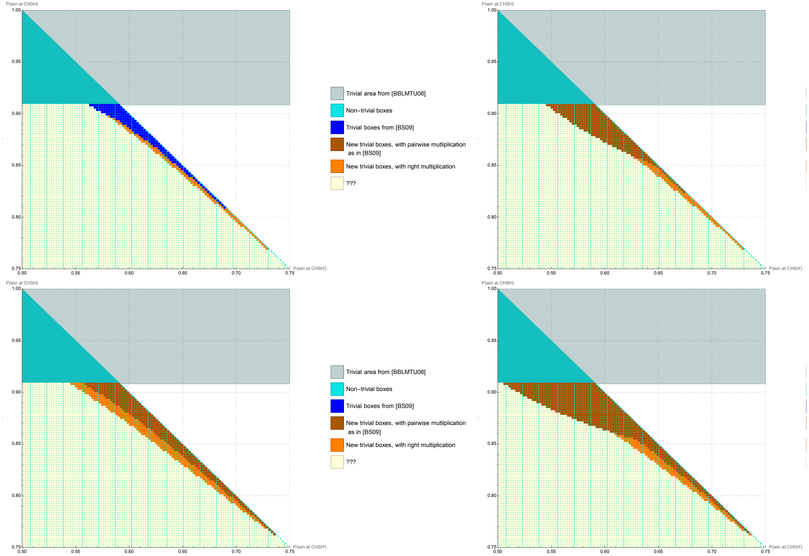
Examples. • $\texttt{Orbit}_3(\mathtt{P}) = \Big\{\mathtt{P} \boxtimes (\mathtt{P} \boxtimes \mathtt{P}),\ (\mathtt{P} \boxtimes \mathtt{P}) \boxtimes \mathtt{P}\Big\}$,

• $\texttt{Orbit}_4(\mathtt{P}) = \Big\{\mathtt{P}\boxtimes\Big(\mathtt{P}\boxtimes(\mathtt{P}\boxtimes\mathtt{P})\Big),\ \mathtt{P}\boxtimes\Big((\mathtt{P}\boxtimes\mathtt{P})\boxtimes\mathtt{P}\Big),\ \Big(\mathtt{P}\boxtimes(\mathtt{P}\boxtimes\mathtt{P})\Big)\boxtimes\mathtt{P},\ \Big((\mathtt{P}\boxtimes\mathtt{P})\boxtimes\mathtt{P}\Big)\boxtimes\mathtt{P},\ \Big(\mathtt{P}\boxtimes\mathtt{P}\Big)\boxtimes\Big(\mathtt{P}\boxtimes\mathtt{P}\Big)\Big\}$.



The "highest" box in each orbit. $\mathtt{P}_{\max,\,k} \;=\; \Big(\Big(\Big(\mathtt{P}\boxtimes\mathtt{P}\Big)\boxtimes\mathtt{P}\Big)\cdots\Big)\boxtimes\mathtt{P} \;=:\; \mathtt{P}^{\boxtimes k}$.

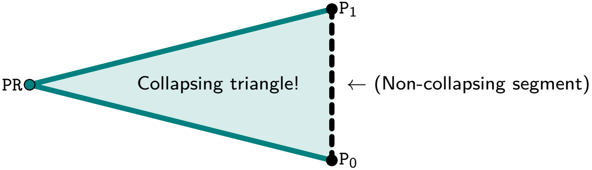
New collapsing boxes: numerically



New collapsing boxes: analytically

Theorem 1 (New collapsing boxes)

In the triangle whose vertices are $\{\mathtt{PR},\ \mathtt{P}_0 := \mathbf{1}_{a=b=0},\ \mathtt{P}_1 := \mathbf{1}_{a=b=1}\}$, *all the points are collapsing boxes, except points in the segment* $\mathtt{P}_0$-$\mathtt{P}_1$.



Proof. **(1)** The triangle is stable under ⊠. **(2)** Define a sequence: initialize at an arbitrary point of the triangle (except in the vertical segment), and inductively apply the multiplication ⊠. **(3)** This sequence converges to PR. **(4)** But, near PR, all boxes are collapsing (cf. 2006). **(5)** Hence, the orbit intersects the collapsing area and the starting box must be collapsing as well. □

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[6] P. Botteron, "Nonlocal boxes and communication complexity," Master's thesis, Université Paul Sabatier (Toulouse), 2022.
Under the supervision of Anne Broadbent, Ion Nechita and Clément Pellegrini.