NonLocal Boxes & Communication Complexity

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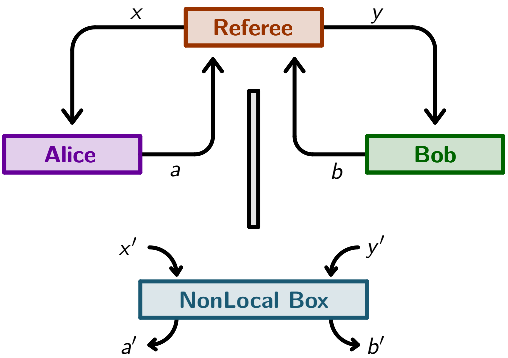
— *Part* 1 —

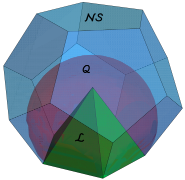
Definitions & Notations

CHSH Game

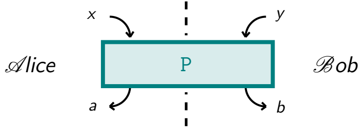


Win at CHSH. $a \oplus b = x\,y$.
Win at CHSH'. $a \oplus b = (x \oplus 1)\,(y \oplus 1)$.

- **Deterministic strategies.**
  $\rightsquigarrow \max \mathbb{P}(\text{win}) = 75\%$.
- **Classical strategies $\mathcal{L}$.**
  $\rightsquigarrow \max \mathbb{P}(\text{win}) = 75\%$.
- **Quantum strategies $\mathcal{Q}$.**
  $\rightsquigarrow \max \mathbb{P}(\text{win}) = \cos^2\left(\frac{\pi}{8}\right) \approx 85\%$.
- **Non-signalling strategies $\mathcal{NS}$.**
  $\rightsquigarrow \max \mathbb{P}(\text{win}) = 100\%$.



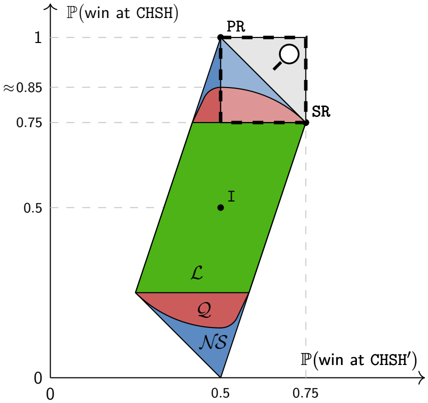
NonLocal Boxes



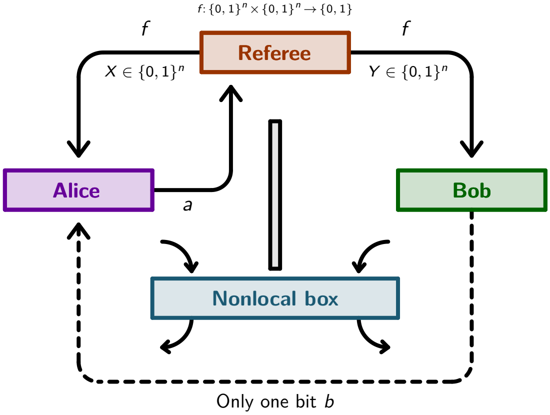
Definition. • *A* **box** *is a conditional probability distribution* $\mathtt{P}\big(a, b \,\big|\, x, y\big)$ *such that* $\mathtt{P} \in \mathcal{NS}$*.*
• *A box* $\mathtt{P}$ *is* **nonlocal** *if* $\mathtt{P} \notin \mathcal{L}$*.*

Examples.

- $\mathtt{PR}\big(a, b \,|\, x, y\big) := \begin{cases} \frac{1}{2} & \text{si } a \oplus b = x\,y, \\ 0 & \text{otherwise.} \end{cases}$
- $\mathtt{SR}\big(a, b \,|\, x, y\big) := \begin{cases} \frac{1}{2} & \text{si } a = b, \\ 0 & \text{otherwise.} \end{cases}$
- $\mathtt{I}\big(a, b \,|\, x, y\big) := \frac{1}{4}.$



Communication Complexity



Win $\iff a = f(X, Y)$.

Def. *A function f is said to be* **trivial** *(in the sense of communication complexity) if Alice knows any value $f(X, Y)$ with only one bit transmitted between Alice and Bob.*

Ex. For $n = 2$, $X = (x_1, x_2)$, $Y = (y_1, y_2)$:

- $f := x_1 \oplus y_1 \oplus x_2 \oplus y_2 \oplus 1$ is trivial.
- $g := (x_1\, x_2) \oplus (y_1\, y_2)$ is trivial.
- $h := (x_1\, y_1) \oplus (x_2\, y_2)$ is NOT trivial.

Def. *A box* P *is said to be* **collapsing** *(or* trivial*) if using copies of this box* P *any Boolean function f is trivial, with probability $\geq q > \frac{1}{2}$.*

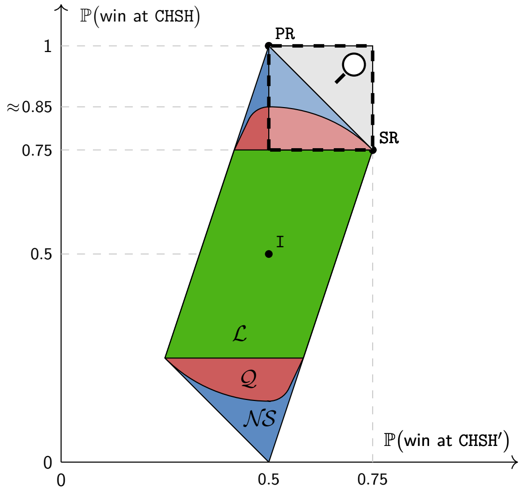
Ex. Link with previous boxes:

- The PR box is collapsing.
- The boxes SR and I are NOT collapsing.

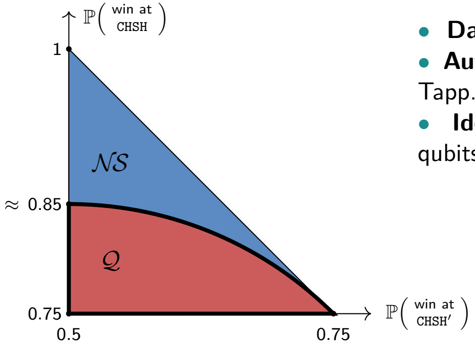
— Part 2 —

Historical Overview

Goal. Show that quantum boxes are **non-collapsing** but that post-quantum boxes are **collapsing**.

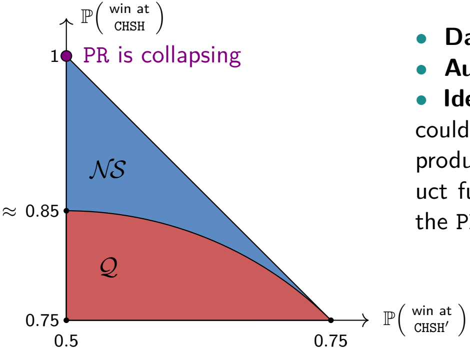


1999: Quantum Boxes are Non-Collapsing



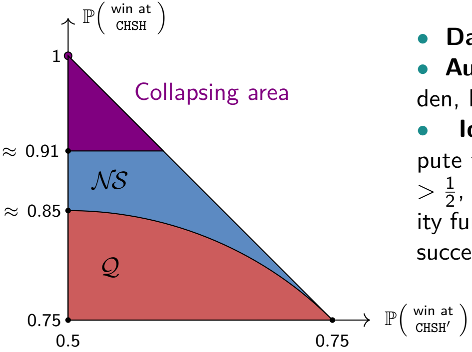
- **Date.** 1999 [1].
- **Authors.** Cleve, van Dam, Nielson, Tapp.
- **Ideas.** (1) Prove the result with qubits, (2) Go back to bits.

1999: The PR Box is Collapsing



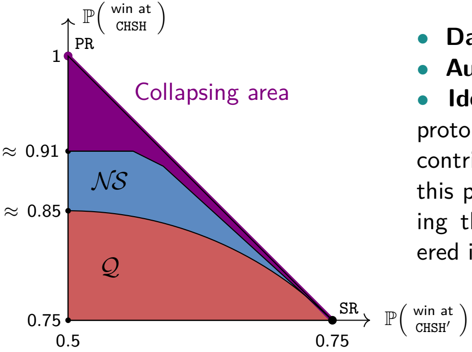
- **Date.** 1999 [2].
- **Author.** van Dam.
- **Ideas.** (1) Any Boolean function f could be written in terms of an inner product function, (2) Any inner product function is trivial using copies of the PR box.

2006: Boxes Above $\approx 91\%$ are Collapsing



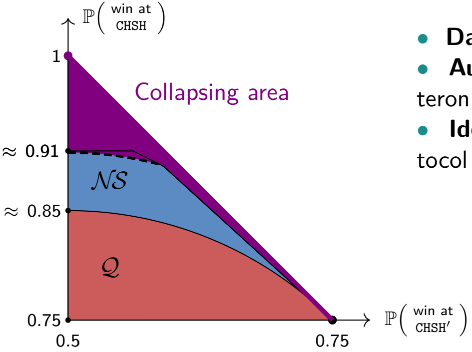
- **Date.** 2006 [3].
- **Authors.** Brassard, Buhrman, Linden, Méthot, Tapp, Unger.
- **Ideas.** (1) Distributively compute the given function f with proba $> \frac{1}{2}$, (2) Inductively apply the majority function `Maj` in order to boost the success probability.

2009: Correlated Boxes are Collapsing



- **Date.** 2009 [4].
- **Authors.** Brunner, Skrzypczyk.
- **Ideas.** (1) Introduce a distillation protocol, cf. generalization in “Our contribution”, (2) Inductively apply this protocol many times until reaching the "collapsing triangle" discovered in 2006.

2023: Boxes above a certain Ellipse are Collapsing

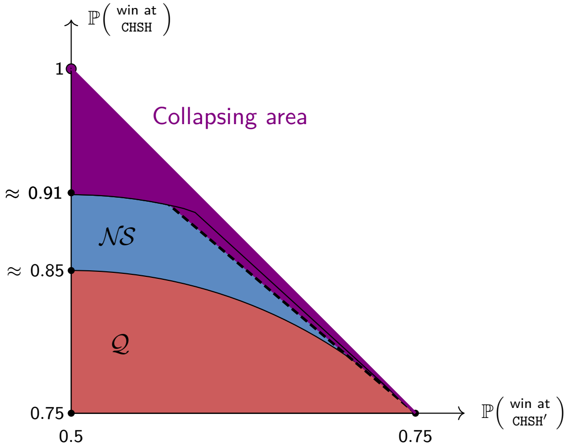


- **Date.** 2023 [5].
- **Author.** Proulx, Broadbent, Botteron.
- **Idea.** Generalize BBLMTU's protocol (cf. 2006).

— *Part* 3 —

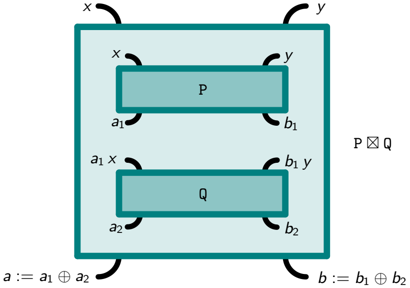
Our Contribution: Algebra of Boxes

Our Contribution [6]



Algebra of Boxes

Recall. *A* nonlocal box P *is a conditional probability distribution* $(a, b, x, y) \in \{0,1\}^4 \mapsto \mathtt{P}(a, b \,|\, x, y) \in [0,1]$ *such that* $\mathtt{P} \in \mathcal{NS} \backslash \mathcal{L}$.



$$\mathtt{P} \boxtimes \mathtt{Q}\left(a,\, b \,\middle|\, x,\, y\right) := \sum_{a_1, b_1 \in \{0,1\}} \mathtt{P}\left(a_1,\, b_1 \,\middle|\, x,\, y\right) \times \mathtt{Q}\left(a \oplus a_1,\, b \oplus b_1 \,\middle|\, a_1 x,\, b_1 y\right)$$

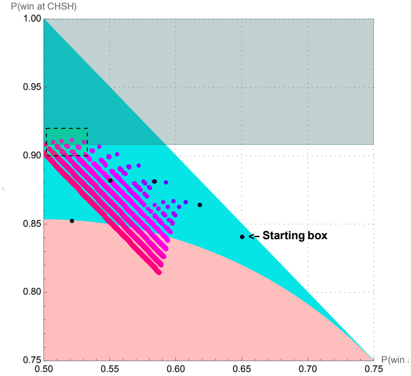
Algebra of Boxes. The vector space $\mathcal{B} := \mathcal{F}\left(\{0,1\}^4, \mathbb{R}\right)$ endowed with the operations $\{+, \cdot, \boxtimes\}$ defines a non-commutative and non-associative algebra.

Orbit of a Box

Orbit of order k. $\texttt{Orbit}_k(\mathtt{P}) := \{\text{products of exactly } k \text{ times the term } \mathtt{P}\}$.

Examples.

- $\texttt{Orbit}_3(\mathtt{P}) = \{\mathtt{P} \boxtimes (\mathtt{P} \boxtimes \mathtt{P}),\ (\mathtt{P} \boxtimes \mathtt{P}) \boxtimes \mathtt{P}\}$,
- $\texttt{Orbit}_4(\mathtt{P}) = \Big\{\mathtt{P}\boxtimes\big(\mathtt{P}\boxtimes(\mathtt{P}\boxtimes\mathtt{P})\big),\ \mathtt{P}\boxtimes\big((\mathtt{P}\boxtimes\mathtt{P})\boxtimes\mathtt{P}\big),\ \big(\mathtt{P}\boxtimes(\mathtt{P}\boxtimes\mathtt{P})\big)\boxtimes\mathtt{P},\ \big((\mathtt{P}\boxtimes\mathtt{P})\boxtimes\mathtt{P}\big)\boxtimes\mathtt{P},\ \big(\mathtt{P}\boxtimes\mathtt{P}\big)\boxtimes\big(\mathtt{P}\boxtimes\mathtt{P}\big)\Big\}$.



Trivial Area

Post–Quantum Correlations

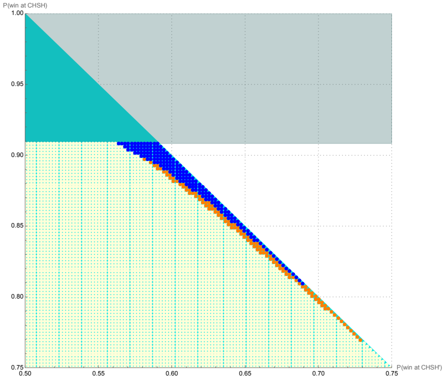
Quantum Correlations

This orbit intersects the trivial area, so this starting box is trivial!

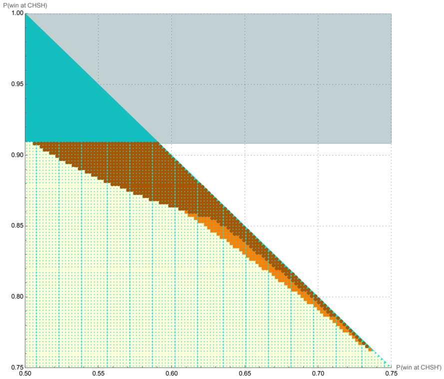
The "highest" box in each orbit.

$$\mathtt{P}_{\max,\,k} = \Big(\big((\mathtt{P} \boxtimes \mathtt{P}) \boxtimes \mathtt{P}\big) \cdots\Big) \boxtimes \mathtt{P} =: \mathtt{P}^{\boxtimes k}.$$

New Collapsing Boxes: Numerical Proof



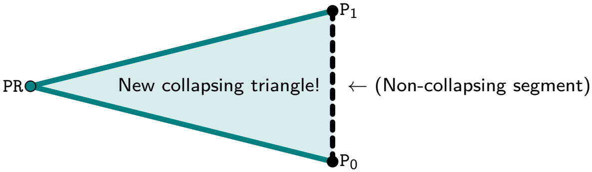
Trivial area from [BBLMTU06]
Non−trivial boxes
Trivial boxes from [BS09]
New trivial boxes, with pairwise mutliplication as in [BS09]
New trivial boxes, with right multiplication
???



New Collapsing Boxes: Analytical Proof

Theorem 1 (New collapsing boxes)

In the triangle whose vertices are $\{\mathtt{PR},\ \mathtt{P}_0 := \mathbf{1}_{a=b=0},\ \mathtt{P}_1 := \mathbf{1}_{a=b=1}\}$*, all the points are collapsing boxes, except points in the segment* $\mathtt{P}_0$*-*$\mathtt{P}_1$*.*



Proof (idea). **(1)** The triangle is stable under ⊠. **(2)** Define a sequence: initialize at an arbitrary point of the triangle (except in the vertical segment), and inductively apply the multiplication ⊠. **(3)** This sequence converges to $\mathtt{PR}$. **(4)** But, near $\mathtt{PR}$, all boxes are collapsing (cf. 2006). **(5)** Hence, the orbit intersects the collapsing area and the starting box must be collapsing as well. □

Bibliography

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[3] G. Brassard, H. Buhrman, N. Linden, A. A. Méthot, A. Tapp, and F. Unger, "Limit on nonlocality in any world in which communication complexity is not trivial," *Phys. Rev. Lett.*, vol. 96, p. 250401, Jun 2006.

[4] N. Brunner and P. Skrzypczyk, "Nonlocality distillation and postquantum theories with trivial communication complexity," *Physical Review Letters*, vol. 102, Apr 2009.

[5] M.-O. Proulx, A. Broadbent, and P. Botteron, "Extending the known region of nonlocal boxes that collapse communication complexity," 2023.

[6] P. Botteron, "Nonlocal boxes and communication complexity," Master's thesis, Université Paul Sabatier (Toulouse), 2022.
Under the supervision of Anne Broadbent, Ion Nechita and Clément Pellegrini.