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MEASUREMENT-PROTECTED QUANTUM PHASES

↑ Title of the article

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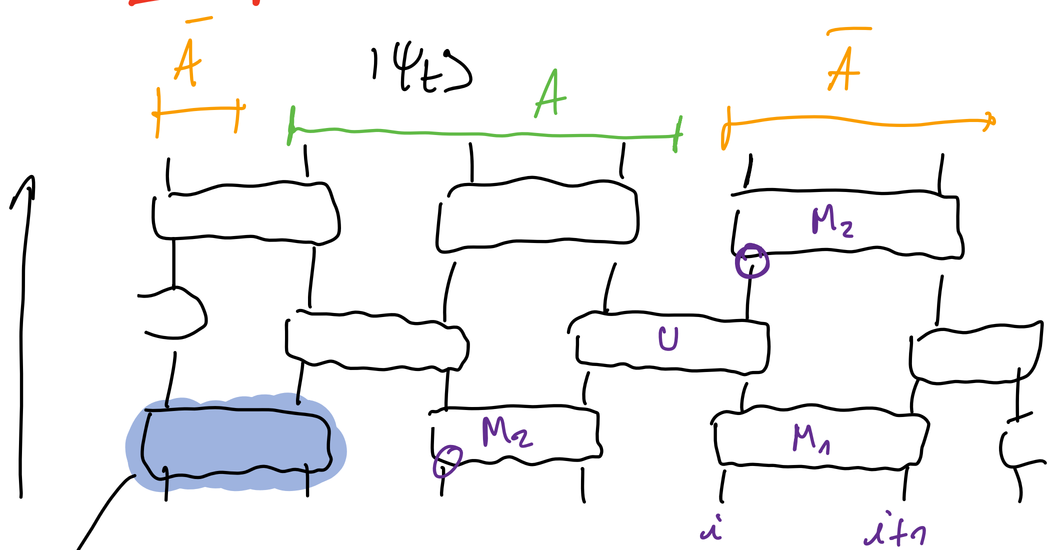
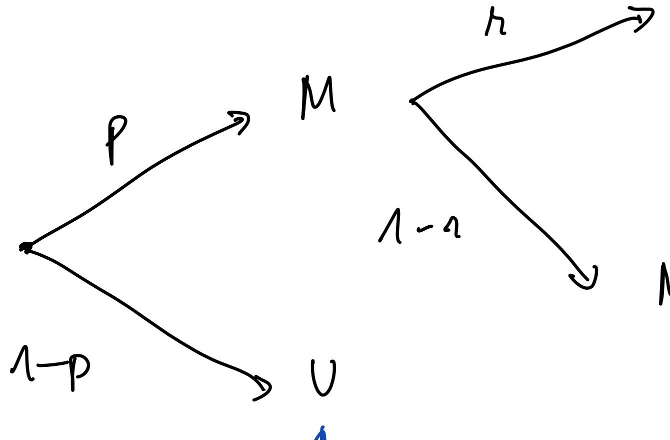
① Setup



$t=0$: $\quad |\psi_0\rangle = |+\rangle \otimes \dots \otimes |+\rangle = |+\rangle^{\otimes L}$

(typically, they take $L = 768$)



M_1 := measurement in the zz-basis $\{|0\rangle, |1\rangle\}$ of the qubits (i) & (i+1)

M_2 := measurement in the x-basis $\{|+\rangle, |-\rangle\}$ of the qubit (i)

U is a Clifford gate (i.e. its sends Pauli strings to Pauli strings), which preserves the Ising symmetry:

$$U^\dagger \left(X \otimes \cdots \otimes X \right) U = X \otimes \cdots \otimes X.$$

Goal: Study the long-time range evolution of $|\psi_t\rangle$

What we expect:

(i) $p \ll 0.5 \Rightarrow$ Unitary evolution

$\Rightarrow$ entanglement growth $\Rightarrow$ "volume-law"

(ii) $p \gg 0.5 \Rightarrow$ many measurements

$\Rightarrow$ disentanglement $\Rightarrow$ "area-law"

Recall: • The Rényi entanglement entropy of $|\psi\rangle$:

$$S_A(|\psi\rangle) = -\log \mathrm{Tr}\left[\rho_A^2\right],$$

where

$$\rho_A = \mathrm{Tr}_{\bar{A}}\left[|\psi\rangle\langle\psi| \right]$$

- Volume law: $S_A \propto |A|$
- Area law: $S_A \propto |\partial A|$

② The Spin-Glass Order Parameter

def: $\mathcal{O}\left(|\psi\rangle \in (\mathbb{C}^2)^{\otimes L}\right) := \frac{1}{L}\sum_{i,j=1}^{L} \langle\psi|Z_i Z_j|\psi\rangle^2 - \underbrace{\langle\psi|Z_i|\psi\rangle^2\langle\psi|Z_j|\psi\rangle^2}_{=0 \text{ by Ising symmetry}}$

where $Z_i := I \otimes \cdots \otimes I \otimes Z \otimes I \otimes \cdots \otimes I$

$\uparrow$ i-th coordinate

Why is it useful? It helps to understand long-time range entanglement:

(i) If $|\psi\rangle = |\psi_1\rangle \otimes \cdots \otimes |\psi_L\rangle$ product state, then $\mathcal{O}(|\psi\rangle)$ is constant in L

Indeed: $|\psi\rangle = |\psi_1\rangle \otimes \cdots \otimes |\psi_L\rangle$

$$\langle\psi|Z_1 Z_2|\psi\rangle = \langle\psi_1|Z|\psi_1\rangle\langle\psi_2|Z|\psi_2\rangle \underbrace{\langle\psi_3|\psi_3\rangle}_{=1}\cdots\underbrace{\langle\psi_L|\psi_L\rangle}_{=1}$$

$$= \langle\psi|Z_1|\psi\rangle\langle\psi|Z_2|\psi\rangle$$

$\forall i \neq j$: $\langle\psi|Z_i Z_j|\psi\rangle = \underbrace{\langle\psi|Z_i|\psi\rangle\langle\psi|Z_j|\psi\rangle}_{=0}$

For $i=j$: $\langle\psi|Z_i Z_i|\psi\rangle = \langle\psi|\psi\rangle = 1$

$$\Rightarrow \mathcal{O}(|\psi\rangle) = \frac{1}{L}\sum_{i,j=1}^{L}\langle\psi|Z_i Z_j|\psi\rangle^2$$

$$= \frac{1}{L} \times L = 1$$

(ii) If $|\Psi\rangle = $ GHZ - state $= \frac{|0\ldots0\rangle + |1\ldots1\rangle}{\sqrt{2}}$,
then $O(|\Psi\rangle)$ is linear in L.

$$\langle\Psi|Z_1 Z_2|\Psi\rangle = \frac{1}{2}\Big(\langle 0|\underbrace{Z|0\rangle}_{=|0\rangle} \langle 0|Z|0\rangle \underbrace{\langle 0\ldots0|0\ldots0\rangle}_{=1}$$

$$+ \langle 0|Z|1\rangle\langle 0|Z|1\rangle \underbrace{\langle 0\ldots0|1\ldots1\rangle}_{=0}$$

$$+ \langle 1|Z|0\rangle\langle 1|Z|0\rangle \underbrace{\langle 1\ldots1|0\ldots0\rangle}_{=0}$$

$$+ \langle 1|\underbrace{Z|1\rangle}_{=-|1\rangle}\langle 1|\underbrace{Z|1\rangle}_{=-|1\rangle} \underbrace{\langle 1\ldots1|1\ldots1\rangle}_{=1}\Big)$$

$$= \frac{1}{2}\left(1 + 0 + 0 + 1\right) = 1$$

Similarly $\forall i \neq j$: $\langle\Psi|Z_i Z_j|\Psi\rangle = 1$

For $i = j$: $\langle\Psi|Z_i Z_i|\Psi\rangle = \langle\Psi|\Psi\rangle = 1$

$$\Rightarrow O(|\Psi\rangle) = \frac{1}{L}\sum_{i,j=1}^{L} \langle\Psi|Z_i Z_j|\Psi\rangle^2$$

$$= \frac{1}{L} L^2 \simeq L .$$

$\Rightarrow$ thus the scaling of $O(|\Psi\rangle)$ is helpful to identify the "spin-glass" area-law: when it is linear in L.

③ Analytical Proof of Phase-Transit

Case $p=1$, $\lambda \in [0,1]$:

(no unitaries, only measurements)

- $\boxed{\lambda = 0}$: Only measurements in the x-basis $\{|+\rangle, |-\rangle\}$

  As $|\psi_0\rangle = |+\rangle \otimes \dots \otimes |+\rangle$

  $\Rightarrow$ $|\psi_t\rangle = |+\rangle \otimes \dots \otimes |+\rangle \quad \forall t \geq 0$

  $\Rightarrow$ "paramagnetic" area-law phase

- $\boxed{\lambda = 1}$: Only measurements in the z-basis $\{|0\rangle, |1\rangle\}$

  $\Rightarrow$ The 1st layer of the circuit turns of the $|+\rangle$ in $|0\rangle$ & $|1\rangle$

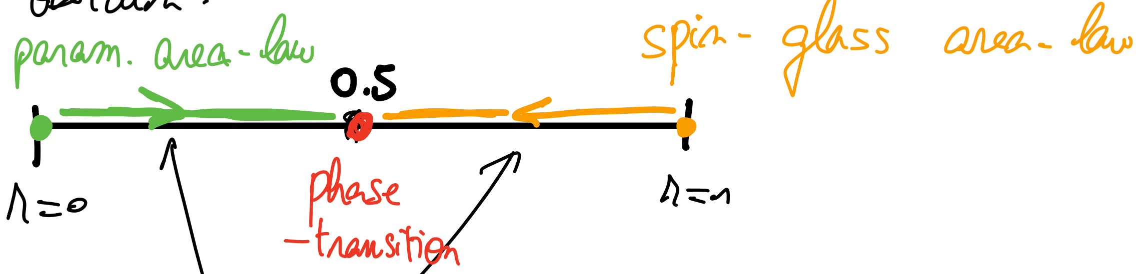
  $\Rightarrow$ $|\psi_t\rangle = |\psi_1\rangle \quad \forall t \geq 1$

  $=$ product state

  $\Rightarrow$ "spin-glass" area-law phase

⇒ We obtain:

$p=1$



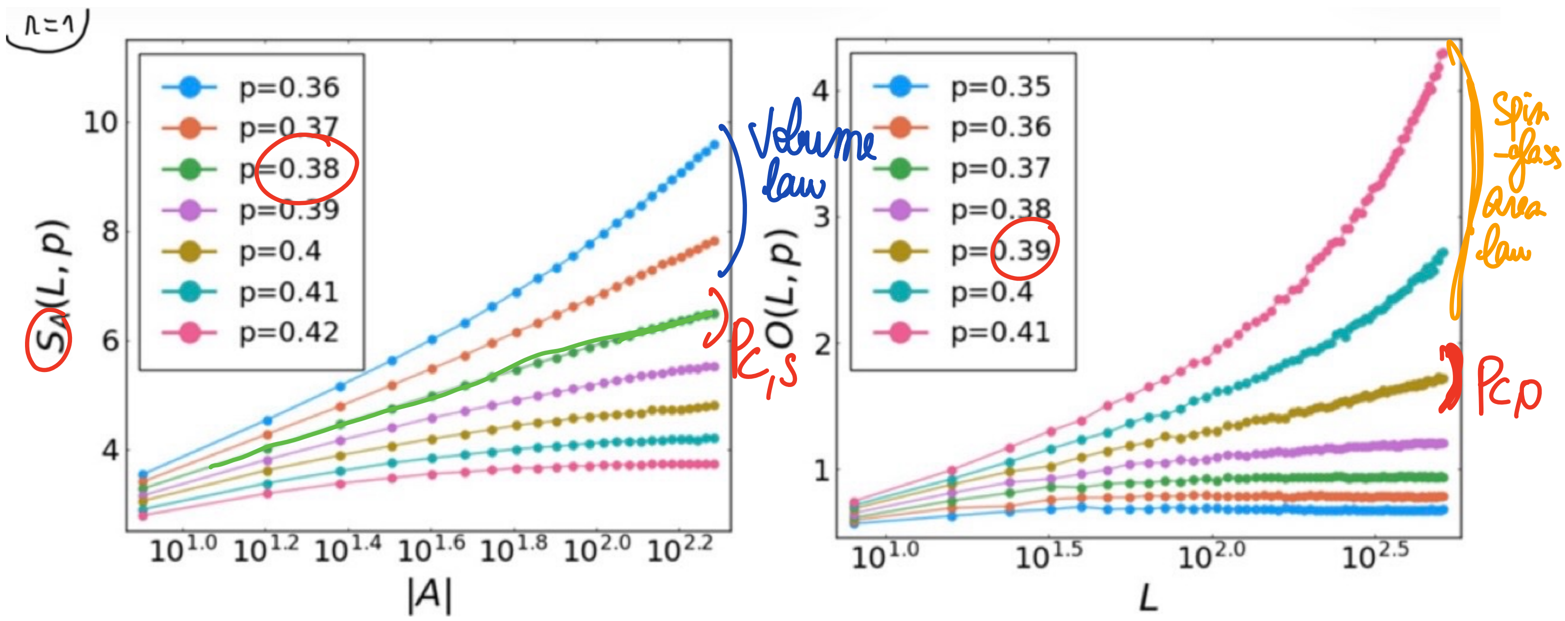
we can extend because the two phases are "perturbatively stable to competing measurements respecting the Ising symmetry"

◦ $\boxed{r=0.5}$ Arguments:

This ensemble of measurement-only circuits has a duality between ZZ, X measurements that is manifest after performing a Jordan-Wigner mapping from spins to Majorana modes. In the latter representation, each spin corresponds to two Majorana modes, and in the resulting Majorana chain, the two types of measurement correspond to fermion parity measurements between pairs of Majoranas on even and odd bonds [Fig. 2(a)]. The duality fixes a phase transition between spin-glass and paramagnetic phases at $r = 0.5$, and this critical point in the Majorana representation is explicitly described by a 2D classical loop model at its corresponding critical point; this mapping was detailed in [32]. See Fig. 2(a) for an example of loops arising from Majorana worldlines.

4 Numerical Proofs of Phase-Transitions

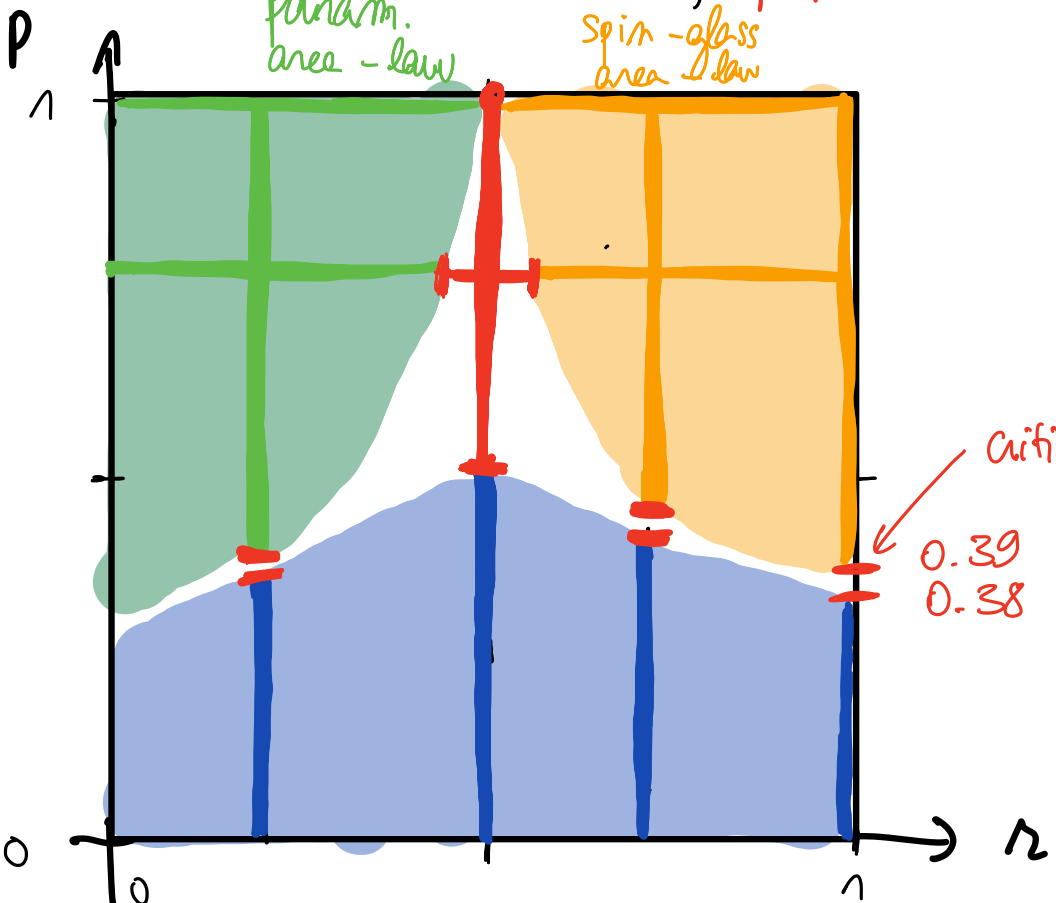
1 $r = 1$, $p \in [0,1]$:



$$\begin{cases} p < p_{c,S} : \text{ volume-law} \\ p_{c,S} \approx 0.38 \end{cases}$$

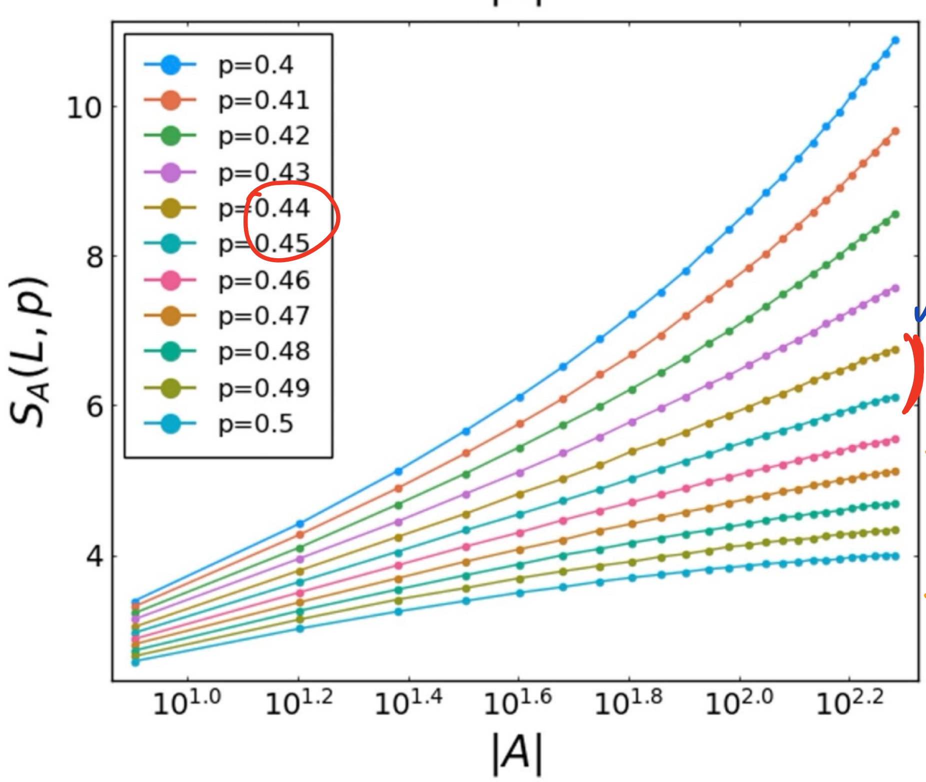
$$\begin{cases} p > p_{c,O} : \text{ spin-glass area-law} \\ p_{c,O} \approx 0.39 \end{cases}$$

Phase diagram



② $r = 0.75$ & $p \in [0,1]$:

$r = 0.75$



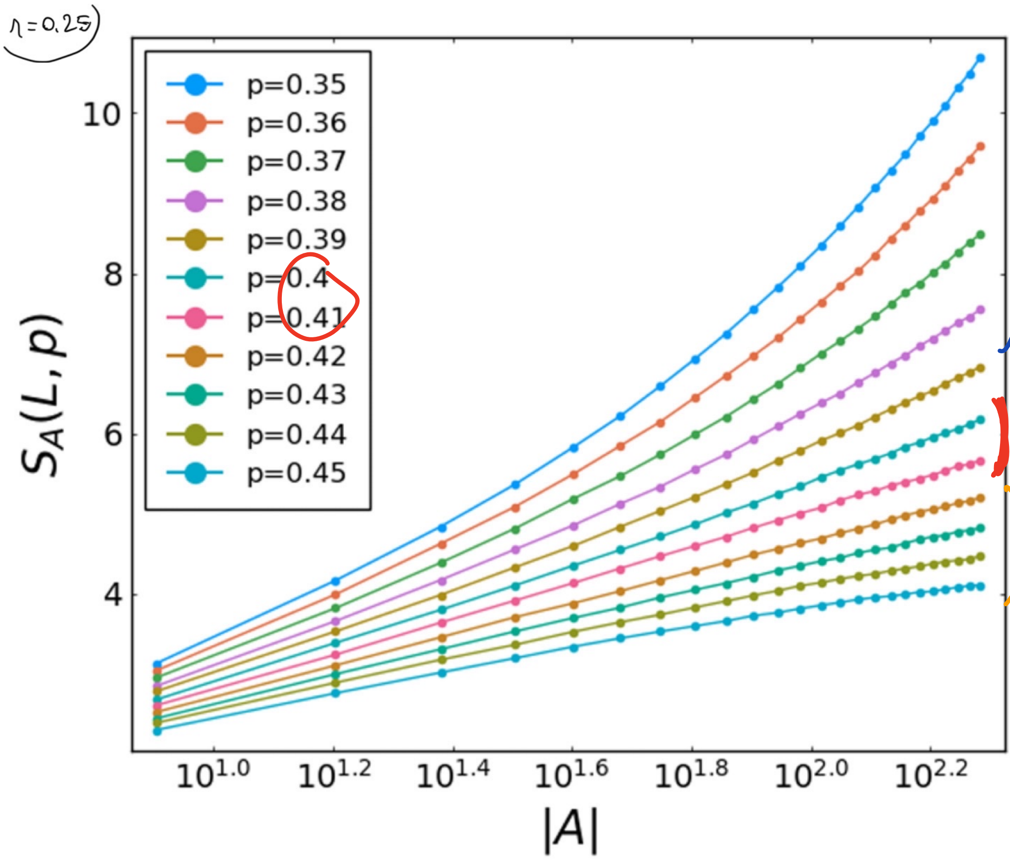
Volume - law

$p_c \approx 0.44, 0.45$

area - law

③ $r = 0.25$:

$r = 0.25$



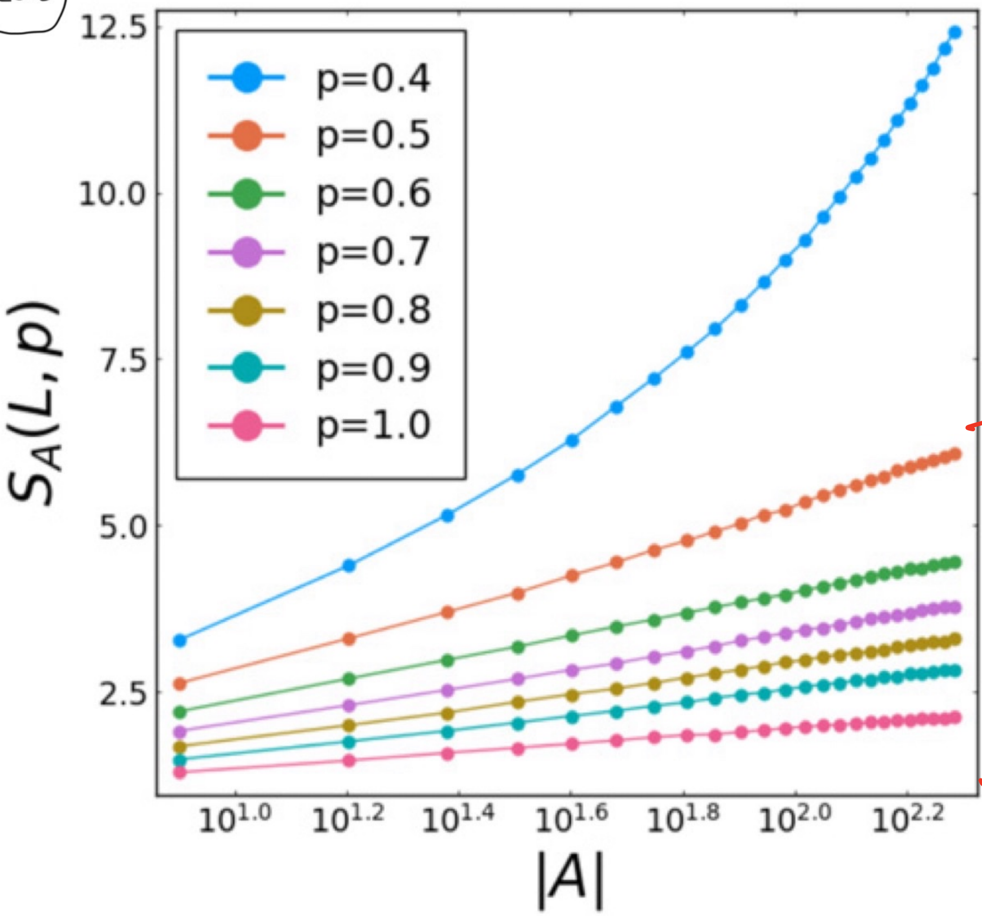
Volume - law

$p_c \approx 0.4, 0.41$

area - law

④ $r = 0.5$:

$r=0.5$

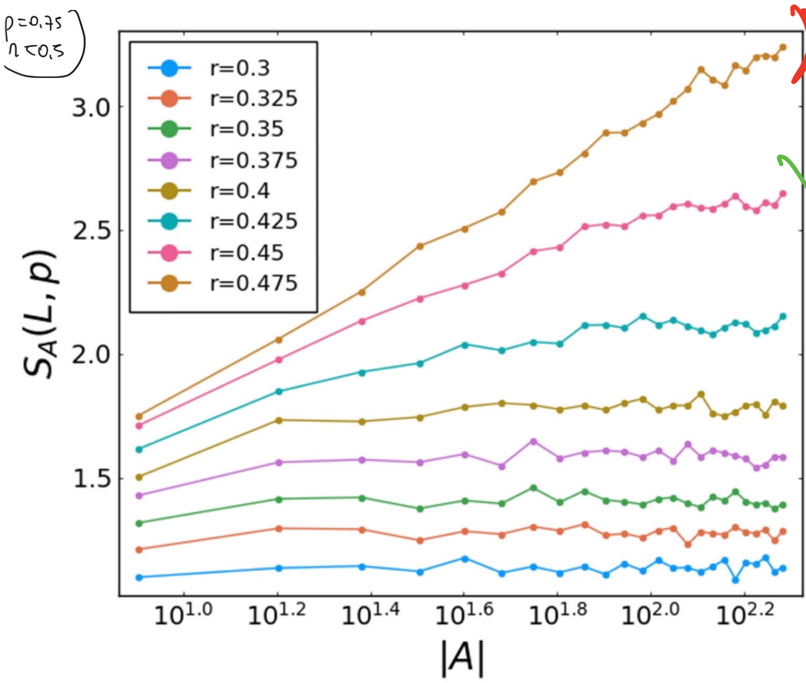


Volume-law

critical points for $p > 0.5$

⑤ $p = 0.75$:

$p=0.75$
$r<0.5$

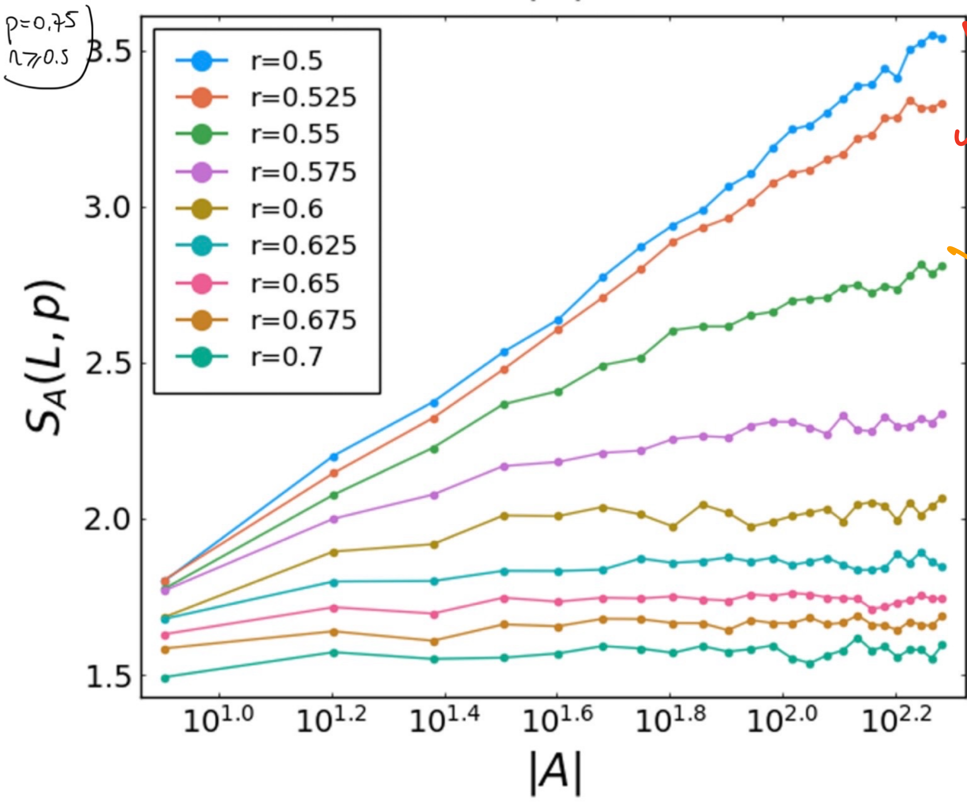


critical

$r < 0.5$:

paramagnetic area-law
$r < 0.475$

p=0.75
r≥0.5

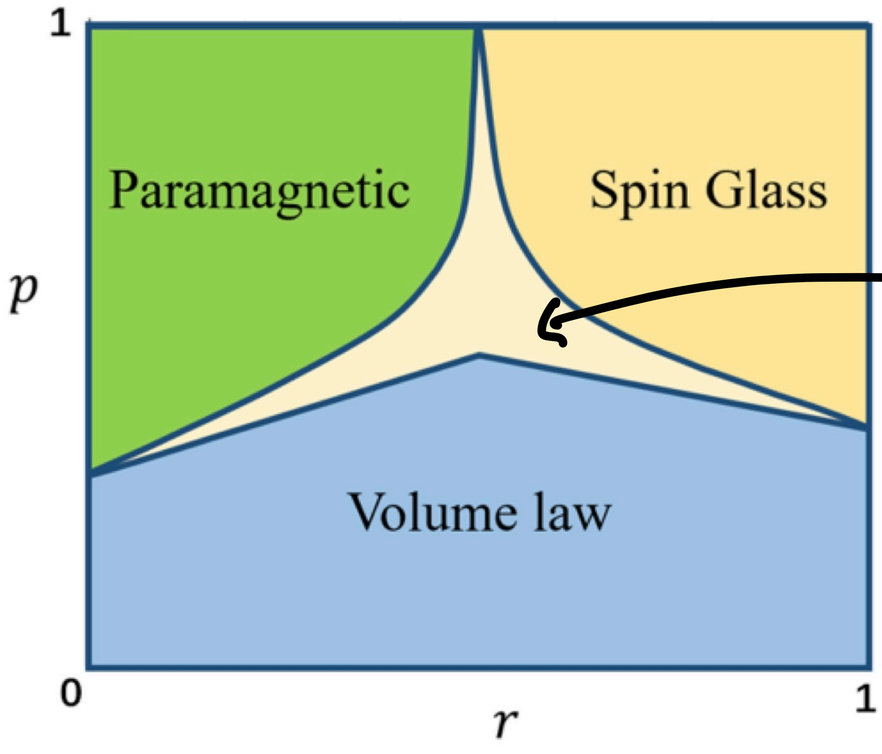


critical

r≳0.5

spin-glass area law for r>0.525

Conclusion:



the critical phase might be "reduced" if we do simulation with L>768