Graph Games & the Collapse of Communication Complexity

Reference: arXiv:2406.02199 [1].

Pierre Botteron
(Toulouse & Ottawa)

and

Moritz Weber
(Saarbrücken)

Ottawa, Friday, September 13^{th}, 2024

Motivation

Goal

Have an information-based description of quantum correlations ($\mathcal{Q}$).

Idea

- Take a larger set than $\mathcal{Q}$: the non-signalling correlations ($\mathcal{NS}$).
- Consider an information-based principle: Communication Complexity (CC).
- Prove that post-quantum correlations ($\mathcal{NS} \setminus \mathcal{Q}$) violate this principle.

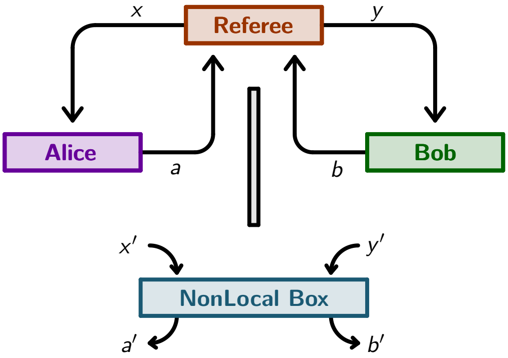
Open Question

What are all non-signalling correlations that violate the principle of CC?

— *Part* 1 —

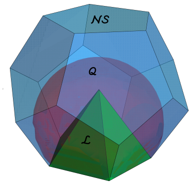
Background

1.1. CHSH Game & Nonlocal Boxes

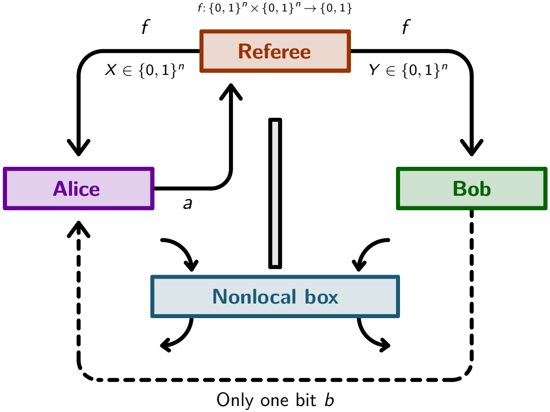


Win at CHSH $\iff a \oplus b = x\,y$.

- **Deterministic Strategies.**
  $\rightsquigarrow \max \mathbb{P}(\text{win}) = 75\%$.
- **Classical Strategies ($\mathcal{L}$).**
  $\rightsquigarrow \max \mathbb{P}(\text{win}) = 75\%$.
- **Quantum Strategies ($\mathcal{Q}$).**
  $\rightsquigarrow \max \mathbb{P}(\text{win}) = \cos^2\left(\frac{\pi}{8}\right) \approx 85\%$.
- **Non-signalling Strategies ($\mathcal{NS}$).**
  $\rightsquigarrow \max \mathbb{P}(\text{win}) = 100\%$.



1.2. Collapse of Communication Complexity



Win $\iff a = f(X, Y)$.

Def. *A function f is said to be* **trivial** *(in the sense of communication complexity) if Alice knows any value $f(X, Y)$ with only one bit transmitted from Bob to Alice.*

Ex. For $n = 2$, $X = (x_1, x_2)$, $Y = (y_1, y_2)$:

- $f := x_1 \oplus y_1 \oplus x_2 \oplus y_2 \oplus 1$ is trivial.
- $g := (x_1\, x_2) \oplus (y_1\, y_2)$ is trivial.
- $h := (x_1\, y_1) \oplus (x_2\, y_2)$ is NOT trivial.

Def. *A box* $\mathtt{P}$ *is said to be* **collapsing** *(or* trivial*) if using copies of this box* $\mathtt{P}$ *any Boolean function f is trivial, with probability $\geqslant q > \frac{1}{2}$ for some q independent of n, f, X, Y.*

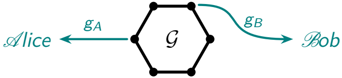
Ex.
- The famous $\mathtt{PR}$ box is collapsing.
- Local ($\mathcal{L}$) and quantum ($\mathcal{Q}$) boxes are NOT collapsing.

— Part 2 —

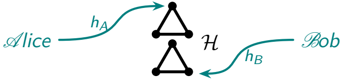
Graph Isomorphism Game

2.1. Definition of the Graph Isomorphism Game

Alice and Bob receive a vertex from a graph $\mathcal{G}$:



and they answer a vertex from a graph $\mathcal{H}$:



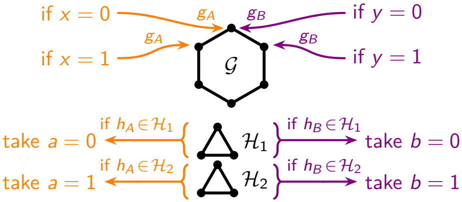
They win the game if and only if:

- $g_A = g_B \Rightarrow h_A = h_B$;
- $g_A \sim g_B \Rightarrow h_A \sim h_B$;
- $g_A \not\simeq g_B \Rightarrow h_A \not\simeq h_B$.

Claim

We can use a perfect strategy for this game to generate a PR box.

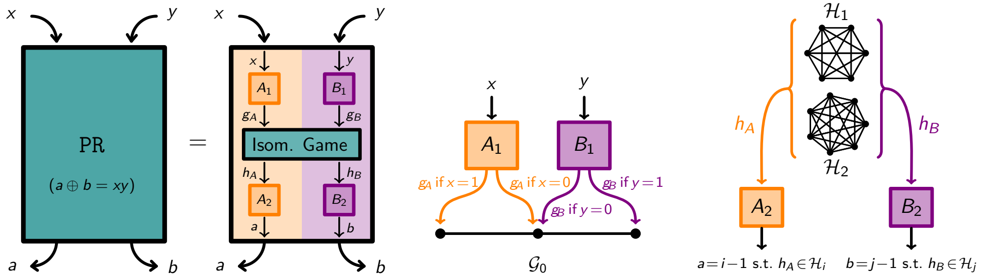
Proof. Let $x, y \in \{0,1\}$. We want to generate $a, b \in \{0,1\}$ such that $a \oplus b = x\,y$.



2.2. Application of Graph Isomorphism to CC

Theorem

Let $\mathcal{G}$ and $\mathcal{H}$ be two graphs such that $\mathrm{diam}(\mathcal{G}) \geqslant 2$, and that $\mathcal{H}$ admits exactly two connected components $\mathcal{H}_1$ and $\mathcal{H}_2$, which are both complete. Then any perfect strategy $\mathcal{S}$ for the graph isomorphism game $\mathcal{G} \cong_{ns} \mathcal{H}$ collapses communication complexity.



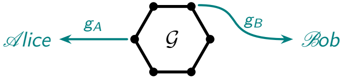
(Other results are available in the manuscript.)

— *Part* 3 —

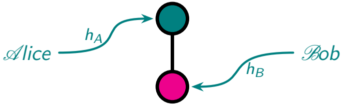
Graph Coloring Game

3.1. Definition of the Graph Coloring Game

Alice and Bob receive a vertex from a graph $\mathcal{G}$:



and they answer a color of their choice:



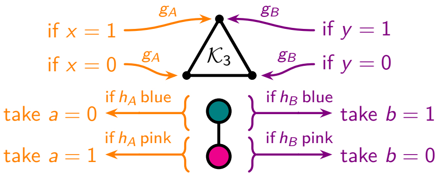
They win the game if and only if:

- $g_A = g_B \Rightarrow h_A = h_B$;
- $g_A \sim g_B \Rightarrow h_A \neq h_B$.

Claim

A perfect strategy for the 2-coloring game of $\mathcal{K}_3$ enables to generate a PR box.

Proof. Let $x, y \in \{0, 1\}$. We want to generate $a, b \in \{0, 1\}$ such that $a \oplus b = x\,y$.



3.2. Application of Graph Coloring to CC

Theorem

Let $\mathcal{G}$ and $\mathcal{H}$ be such that $\operatorname{diam}(\mathcal{G}) \geqslant 2$, and that $\mathcal{H}$ admits exactly N connected components $\mathcal{H}_1, \dots, \mathcal{H}_N$, all being complete. Then, given any strategy $\mathcal{S}$ winning the graph isomorphism game $\mathcal{G} \cong_{ns} \mathcal{H}$ with probability $\mathfrak{p}$, combined with an $\mathcal{NS}$-strategy winning the 2-coloring game of $\mathcal{K}_N$ with probability $\mathfrak{q}$ such that $\mathfrak{p}\mathfrak{q} > \frac{3+\sqrt{6}}{6} \approx 0.91$, there is a collapse of communication complexity.

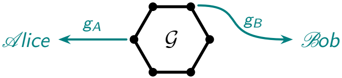
(Other results are available in the manuscript.)

— *Part* 4 —

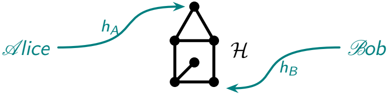
Vertex Distance Game

4.1. Definition of the Vertex Distance Game

Alice and Bob receive a vertex from a graph $\mathcal{G}$:



and they answer a vertex from a graph $\mathcal{H}$:



They win the game if and only if:

$$d(h_A, h_B) = \begin{cases} d(g_A, g_B) & \text{if } d(g_A, g_B) \leqslant D\,, \\ > D & \text{otherwise}\,. \end{cases}$$

If they win for all g_A, g_B, we denote $\mathcal{G} \cong^D \mathcal{H}$.

$$\cdots \Rightarrow \mathcal{G} \cong^{D=2} \mathcal{H} \Rightarrow \mathcal{G} \cong^{D=1} \mathcal{H} \Rightarrow \mathcal{G} \cong^{D=0} \mathcal{H}\,.$$

Interesting Cases

- $D=0$: Graph Bisynchronous Game.
- $D=1$: Graph Isomorphism Game.
- $D=\mathrm{diam}(\mathcal{H})$:

$$d(h_A, h_B) = \begin{cases} d(g_A, g_B) & \text{if } d(g_A, g_B) \leqslant \mathrm{diam}(\mathcal{H}), \\ \infty & \text{otherwise}. \end{cases}$$

Rmk: If $\mathcal{G} \cong^D \mathcal{H}$, then $|V(\mathcal{G})| = |V(\mathcal{H})|$.

4.2. Classical and Quantum Strategies

Perfect classical (resp. quantum) strategies for
the graph isomorphism game ($D = 1$) and for the vertex distance game ($D \geqslant 2$) coincide:

Theorem (Classical Strategies)

Let $D \geqslant 1$. The following are equivalent:

- $\mathcal{G} \cong^D \mathcal{H}$.
- $\mathcal{G} \cong \mathcal{H}$.
- $\exists$ perm. matrix P s.t. $A_\mathcal{G} P = P A_\mathcal{H}$.
- $\forall \mathcal{K}$, $\#\mathrm{Hom}(\mathcal{K},\mathcal{G}) = \#\mathrm{Hom}(\mathcal{K},\mathcal{H})$.
- $\forall \mathcal{K}$, $\#\mathrm{Hom}(\mathcal{G},\mathcal{K}) = \#\mathrm{Hom}(\mathcal{H},\mathcal{K})$.

Theorem (Quantum Strategies)

Let $D \geqslant 1$. The following are equivalent:

- $\mathcal{G} \cong_q^D \mathcal{H}$.
- $\mathcal{G} \cong_q \mathcal{H}$.
- $\exists$ quantum permutation matrix P s.t. $A_\mathcal{G} P = P A_\mathcal{H}$.
- $\forall \mathcal{K}$ planar, $\#\mathrm{Hom}(\mathcal{K},\mathcal{G}) = \#\mathrm{Hom}(\mathcal{K},\mathcal{H})$.

4.3. Non-Signalling Strategies

Def. $\mathcal{G} \cong_{frac} \mathcal{H} \iff \exists P$ bistochastic s.t. $A_{\mathcal{G}} P = P A_{\mathcal{H}}$, where $A_{\mathcal{G}}$ is the adjacency matrix, with coefficient 1 for adjacent vertices, and coefficient 0 otherwise.

Theorem (Atserias *et.al.* [2], Ramana *et.al.* [3])

The following are equivalent:

- $\mathcal{G} \cong_{ns} \mathcal{H}$.
- $\mathcal{G} \cong_{frac} \mathcal{H}$.
- $(\mathcal{G}, \mathcal{H})$ admits a common equitable partition.

Def. $\mathcal{G} \cong_{frac}^{D} \mathcal{H} \iff \exists P$ bistochastic s.t. $A_{\mathcal{G}}^{(t)} P = P A_{\mathcal{H}}^{(t)}$ for all $t \leqslant D$, where $A_{\mathcal{G}}^{(t)}$ is the matrix with coefficient 1 for vertices at distance t, and coefficient 0 otherwise.

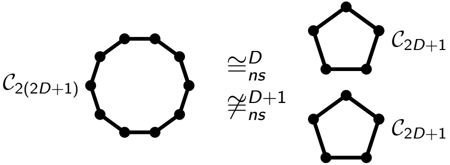
Theorem (Non-Signalling Strategies)

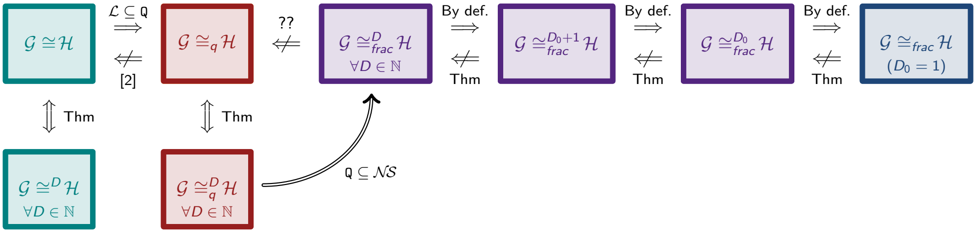
Let $D \geqslant 0$. The following are equivalent:

- $\mathcal{G} \cong_{ns}^{D} \mathcal{H}$.
- $\mathcal{G} \cong_{frac}^{D} \mathcal{H}$.
- $(\mathcal{G}, \mathcal{H})$ admits a D-common equitable partition.

4.4. Strict Implications

As opposed to classical and quantum strategies, perfect $\mathcal{NS}$ strategies do not coincide between the isomorphism game ($D = 1$) and the distance game ($D \geqslant 2$), and more generally:





4.5. Application of Vertex Distance to CC

Theorem

If $\text{diam}(\mathcal{G}) > \text{diam}(\mathcal{H}) \geqslant D \geqslant 1$ and if $\mathcal{H}$ admits exactly two connected components, then any perfect $\mathcal{NS}$-strategy for the D-distance game collapses communication complexity.

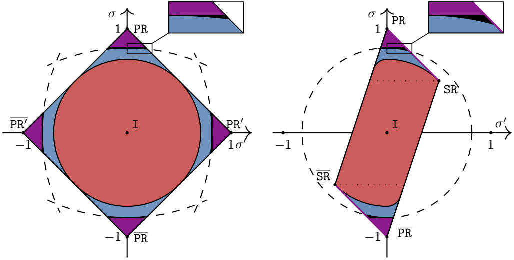
Theorem

If $\text{diam}(\mathcal{G}) > \text{diam}(\mathcal{H}) \geqslant D \geqslant 1$ and if $\mathcal{H}$ admits exactly N connected components, then any $\mathcal{NS}$-strategy winning the D-distance game with probability $\mathfrak{p}$, combined with an $\mathcal{NS}$-strategy winning the 2-coloring game of $\mathcal{K}_N$ with probability $\mathfrak{q}$, such that $\mathfrak{pq} < \frac{3+\sqrt{6}}{6}$, collapses communication complexity.

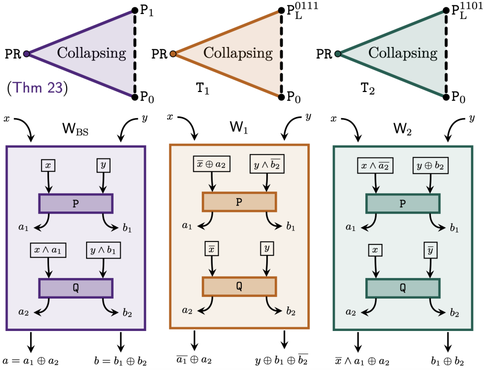
(Other results are available in the manuscript.)

Our Other Related Results

B.–Broadbent–Proulx, PRL:132 (7 2024) [4]. We find that boxes above a certain ellipse collapse CC, using bias amplification by majority function:



B.–Broadbent–Chhaibi–Nechita–Pellegrini, Quantum 8, 1402 (2024) [5]. We study wirings between nonlocal boxes and use them to find boxes that collapse CC:



Bibliography

[1] P. Botteron and M. Weber, "Communication complexity of graph isomorphism, coloring, and distance games," 2024.

[2] A. Atserias, L. Mancinska, D. E. Roberson, R. Samal, S. Severini, and A. Varvitsiotis, "Quantum and non-signalling graph isomorphisms," *Journal of Combinatorial Theory, Series B*, vol. 136, pp. 289–328, 2019.

[3] M. V. Ramana, E. R. Scheinerman, and D. Ullman, "Fractional isomorphism of graphs," 1994.

[4] P. Botteron, A. Broadbent, and M.-O. Proulx, "Extending the known region of nonlocal boxes that collapse communication complexity," *Phys. Rev. Lett.*, vol. 132, p. 070201, Feb 2024.

[5] P. Botteron, A. Broadbent, R. Chhaibi, I. Nechita, and C. Pellegrini, "Algebra of Nonlocal Boxes and the Collapse of Communication Complexity," *Quantum*, vol. 8, p. 1402, July 2024.

[6] P. J. Cameron, A. Montanaro, M. W. Newman, S. Severini, and A. Winter, "On the quantum chromatic number of a graph," 2006.

[7] M. Lupini, L. Mancinska, and D. E. Roberson, "Nonlocal games and quantum permutation groups," *Journal of Functional Analysis*, vol. 279, no. 5, p. 108592, 2020.

[8] L. Lovász, "Operations with structures," 1967.

[9] S. Chaudhuri and M. Y. Vardi, "Optimization of real conjunctive queries," 1993.

[10] L. Mancinska and D. E. Roberson, "Quantum isomorphism is equivalent to equality of homomorphism counts from planar graphs," 2020.

[11] G. Brassard, H. Buhrman, N. Linden, A. A. Méthot, A. Tapp, and F. Unger, "Limit on nonlocality in any world in which communication complexity is not trivial," *Phys. Rev. Lett.*, vol. 96, p. 250401, 06 2006.