Graph Games
& Communication Complexity

Reference: arXiv:2406.02199 [1].

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Motivation

Goal

Have an information-based description of quantum correlations ($\mathcal{Q}$).

Idea

- Take a larger set than $\mathcal{Q}$: the non-signalling correlations ($\mathcal{NS}$).
- Consider an information-based principle: Communication Complexity (CC).
- Prove that quantum correlations satisfy this principle, and that post-quantum correlations ($\mathcal{NS} \setminus \mathcal{Q}$) violate it.

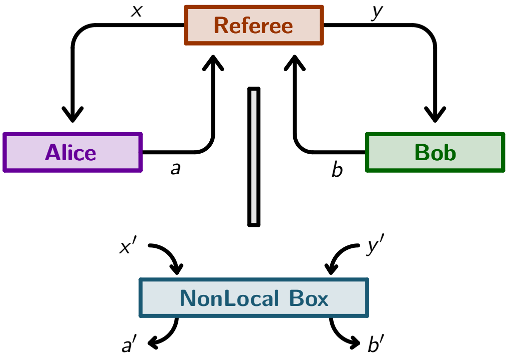
Open Question

What are all non-signalling correlations that violate the principle of CC?

— *Part* 1 —

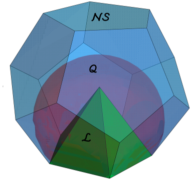
Background

CHSH Game & Nonlocal Boxes

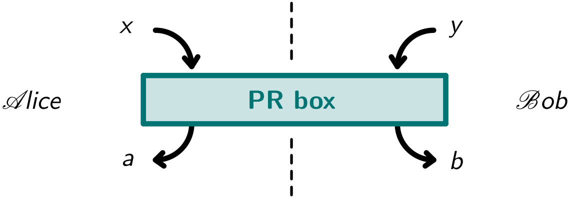


Win at CHSH $\iff$ $a \oplus b = x\,y$.

- **Deterministic Strategies.**
  $\rightsquigarrow$ $\max \mathbb{P}(\text{win}) = 75\%$.
- **Classical Strategies ($\mathcal{L}$).**
  $\rightsquigarrow$ $\max \mathbb{P}(\text{win}) = 75\%$.
- **Quantum Strategies ($\mathcal{Q}$).**
  $\rightsquigarrow$ $\max \mathbb{P}(\text{win}) = \cos^2\left(\frac{\pi}{8}\right) \approx 85\%$.
- **Non-signalling Strategies ($\mathcal{NS}$).**
  $\rightsquigarrow$ $\max \mathbb{P}(\text{win}) = 100\%$.



PR box



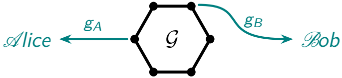
such that $a \oplus b = xy$.

— *Part 2* —

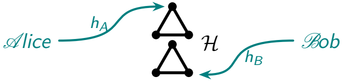
Graph Isomorphism Game

Definition of the Graph Isomorphism Game

Alice and Bob receive a vertex from a graph $\mathcal{G}$:



and they answer a vertex from a graph $\mathcal{H}$:



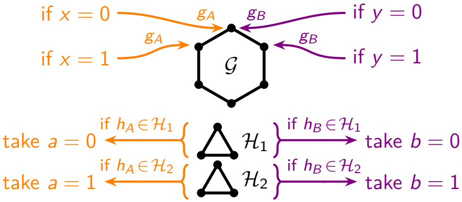
They win the game if and only if:

- $g_A = g_B \Rightarrow h_A = h_B$;
- $g_A \sim g_B \Rightarrow h_A \sim h_B$;
- $g_A \not\simeq g_B \Rightarrow h_A \not\simeq h_B$.

Claim

We can use a perfect strategy for this game to generate a PR box.

Proof. Let $x, y \in \{0,1\}$. We want to generate $a, b \in \{0,1\}$ such that $a \oplus b = x\,y$.



Theorem 1 (B.–Weber)

If $\text{diam}(\mathcal{G}) \geqslant 2$ and if $\mathcal{H} = \mathcal{K}_n \sqcup \mathcal{K}_m$ where $\mathcal{K}_n, \mathcal{K}_m$ are complete graphs, then from *any* perfect strategy for the isomorphism game of $(\mathcal{G}, \mathcal{H})$, one can generate a PR box.

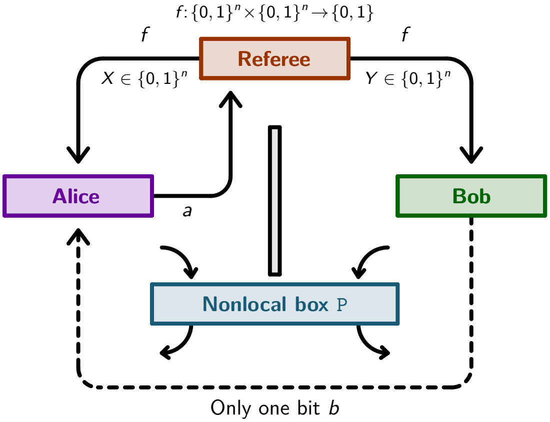
Theorem 2 (B.–Weber)

Let $\mathcal{G} \cong_{ns} \mathcal{H}$ such that $\text{diam}(\mathcal{G}) \geqslant 2$ and $\mathcal{H}$ is not connected. Assume "some symmetry" in a common equitable partition of $(\mathcal{G}, \mathcal{H})$. Then *there exists* a perfect strategy for the isomorphism game of $(\mathcal{G}, \mathcal{H})$ that generates a PR box.

Theorem 3 (B.–Weber)

Let $\mathcal{G}$ and $\mathcal{H}$ be like in Thm 2. Assume moreover that $\mathcal{H}$ is strongly transitive and regular. Then *every* perfect strategy for the isom. game of $(\mathcal{G}, \mathcal{H})$ generates a PR box.

Collapse of Communication Complexity



Win $\iff a = f(X,Y)$.

Def. We say that a nonlocal box P *collapses CC* if $\exists q > 1/2$ such that $\forall n \in \mathbb{N}$, $\forall f : \{0,1\}^{2n} \to \{0,1\}$, and $\forall X, Y \in \{0,1\}^n$, we have:

$$\mathbb{P}\big(a = f(X,Y)\,\big|\,X, Y, \mathrm{P}\big) \geqslant q\,.$$

Fact (van Dam)

The PR box collapses CC.

Corollary (B.–Weber)

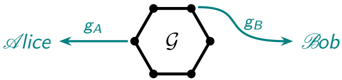
The perfect strategies presented in Thms 1,2,3 for the isomorphism game of $(\mathcal{G},\mathcal{H})$ collapse CC.

— *Part* 3 —

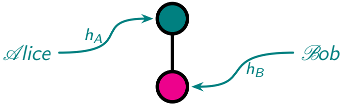
Graph Coloring Game

Definition of the Graph Coloring Game

Alice and Bob receive a vertex from a graph $\mathcal{G}$:



and they answer a color of their choice:



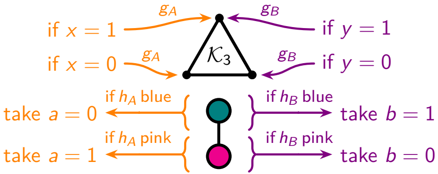
They win the game if and only if:

- $g_A = g_B \Rightarrow h_A = h_B$;
- $g_A \sim g_B \Rightarrow h_A \neq h_B$.

Claim

A perfect strategy for the 2-coloring game of $\mathcal{K}_3$ enables to generate a PR box.

Proof. Let $x, y \in \{0,1\}$. We want to generate $a, b \in \{0,1\}$ such that $a \oplus b = x\,y$.



Application of Graph Coloring to CC

Theorem 4 (B.–Weber)

Let $\mathcal{G}$ and $\mathcal{H}$ be such that $\mathrm{diam}(\mathcal{G}) \geqslant 2$, and that $\mathcal{H}$ admits exactly N connected components $\mathcal{H}_1, \ldots, \mathcal{H}_N$, all being complete. Then, given any strategy winning the graph isomorphism game $\mathcal{G} \cong_{ns} \mathcal{H}$ with probability $\mathfrak{p}$, combined with an $\mathcal{NS}$-strategy winning the 2-coloring game of $\mathcal{K}_N$ with probability $\mathfrak{q}$ such that $\mathfrak{pq} > \frac{3+\sqrt{6}}{6} \approx 0.91$, there is a collapse of communication complexity.

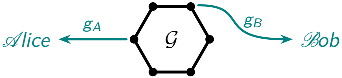
(Other results are available in the manuscript.)

— *Part* 4 —

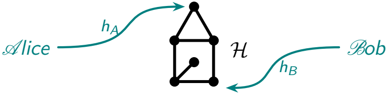
Vertex Distance Game

Definition of the Vertex Distance Game

Alice and Bob receive a vertex from a graph $\mathcal{G}$:



and they answer a vertex from a graph $\mathcal{H}$:



Let $D \in \mathbb{N}$. They win the game if and only if:

$$d(h_A, h_B) = \begin{cases} d(g_A, g_B) & \text{if } d(g_A, g_B) \leqslant D\,, \\ > D & \text{otherwise}\,. \end{cases}$$

If they win for all g_A, g_B, we denote $\mathcal{G} \cong^D \mathcal{H}$.

$$\dots \Rightarrow \mathcal{G} \cong^{D=2} \mathcal{H} \Rightarrow \mathcal{G} \cong^{D=1} \mathcal{H} \Rightarrow \mathcal{G} \cong^{D=0} \mathcal{H}\,.$$

Particular Cases

- $D=0$: Graph Bisynchronous Game.
- $D=1$: Graph Isomorphism Game.
- $D=\mathrm{diam}(\mathcal{H})$:

$$d(h_A, h_B) = \begin{cases} d(g_A, g_B) & \text{if } d(g_A, g_B) \leqslant \mathrm{diam}(\mathcal{H}), \\ \infty & \text{otherwise}. \end{cases}$$

Rmk: If $\mathcal{G} \cong^D \mathcal{H}$, then $|V(\mathcal{G})| = |V(\mathcal{H})|$.

Classical and Quantum Strategies

Perfect classical (resp. quantum) strategies for
the graph isomorphism game ($D = 1$) and for the vertex distance game ($D \geqslant 2$) coincide:

Theorem 5 (B.–Weber)

Let $D \geqslant 1$. The following are equivalent:

- $\mathcal{G} \cong^D \mathcal{H}$;
- $\mathcal{G} \cong \mathcal{H}$;

the latter being equivalent to[1]:

- $\exists$ perm. matrix P s.t. $A_\mathcal{G} P = P A_\mathcal{H}$;
- $\forall \mathcal{K}$, $\#\,\mathrm{Hom}(\mathcal{K}, \mathcal{G}) = \#\,\mathrm{Hom}(\mathcal{K}, \mathcal{H})$;
- $\forall \mathcal{K}$, $\#\,\mathrm{Hom}(\mathcal{G}, \mathcal{K}) = \#\,\mathrm{Hom}(\mathcal{H}, \mathcal{K})$.

Theorem 6 (B.–Weber)

Let $D \geqslant 1$. The following are equivalent:

- $\mathcal{G} \cong_q^D \mathcal{H}$;
- $\mathcal{G} \cong_q \mathcal{H}$;

the latter being equivalent to[2]:

- $\exists$ quantum permutation matrix P s.t. $A_\mathcal{G} P = P A_\mathcal{H}$;
- $\forall \mathcal{K}$ planar, $\#\mathrm{Hom}(\mathcal{K}, \mathcal{G}) = \#\mathrm{Hom}(\mathcal{K}, \mathcal{H})$.

[1] [Lovász'67], [Chaudhuri–Vardi'93]; [2] [Lupini–Mančinska–Roberson'20], [Mančinska–Roberson'20].

Non-Signalling Strategies

Recall. $\mathcal{G} \cong_{frac} \mathcal{H} \iff \exists P$ bistochastic s.t. $A_\mathcal{G} P = P A_\mathcal{H}$, where $A_\mathcal{G}$ is the adjacency matrix, with coefficient 1 for adjacent vertices, and coefficient 0 otherwise.

Theorem (Ramana–Scheinerman–Ullman 1994, Atserias–Mančinska–Roberson–*et.al.* 2019)

The following are equivalent:

- $\mathcal{G} \cong_{ns} \mathcal{H}$.
- $\mathcal{G} \cong_{frac} \mathcal{H}$.
- $(\mathcal{G}, \mathcal{H})$ admits a common equitable partition.

Def. $\mathcal{G} \cong_{frac}^D \mathcal{H} \iff \exists P$ bistochastic s.t. $A_\mathcal{G}^{(t)} P = P A_\mathcal{H}^{(t)}$ for all $t \leqslant D$, where $A_\mathcal{G}^{(t)}$ is the matrix with coefficient 1 for vertices at distance t, and coefficient 0 otherwise.

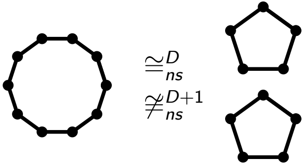
Theorem 7 (B.–Weber)

Let $D \geqslant 0$. The following are equivalent:

- $\mathcal{G} \cong_{ns}^D \mathcal{H}$.
- $\mathcal{G} \cong_{frac}^D \mathcal{H}$.
- $(\mathcal{G}, \mathcal{H})$ admits a D-common equitable partition.

Strict Implications

As opposed to classical and quantum strategies, perfect $\mathcal{NS}$ strategies do not coincide between the isomorphism game ($D = 1$) and the distance game ($D \geqslant 2$), and more generally:



$$
\begin{tikzcd}
\boxed{\mathcal{G} \cong \mathcal{H}} \arrow[r, Rightarrow, shift left, "\mathcal{L} \subseteq \mathtt{Q}"] \arrow[d, Leftrightarrow, "\text{Thm}"] & \boxed{\mathcal{G} \cong_q \mathcal{H}} \arrow[l, "\not\Leftarrow", phantom, shift left, "{[3]}"'] \arrow[d, Leftrightarrow, "\text{Thm}"] & \boxed{\begin{array}{c}\mathcal{G} \cong^{D}_{frac} \mathcal{H} \\ \forall D \in \mathbb{N}\end{array}} \arrow[l, phantom, "\not\Leftarrow"] \arrow[r, Rightarrow, shift left, "\text{By def.}"] & \boxed{\mathcal{G} \cong^{D_0+1}_{frac} \mathcal{H}} \arrow[l, phantom, shift left, "\not\Leftarrow"', "\text{Thm}"'] \arrow[r, Rightarrow, shift left, "\text{By def.}"] & \boxed{\mathcal{G} \cong^{D_0}_{frac} \mathcal{H}} \arrow[l, phantom, shift left, "\not\Leftarrow"', "\text{Thm}"'] \arrow[r, Rightarrow, shift left, "\text{By def.}"] & \boxed{\begin{array}{c}\mathcal{G} \cong_{frac} \mathcal{H} \\ (D_0 = 1)\end{array}} \arrow[l, phantom, shift left, "\not\Leftarrow"', "\text{Thm}"'] \\
\boxed{\begin{array}{c}\mathcal{G} \cong^{D} \mathcal{H} \\ \forall D \in \mathbb{N}\end{array}} & \boxed{\begin{array}{c}\mathcal{G} \cong^{D}_{q} \mathcal{H} \\ \forall D \in \mathbb{N}\end{array}} \arrow[ur, Rightarrow, bend right, "\mathtt{Q} \subseteq \mathcal{NS}"'] & & & &
\end{tikzcd}
$$

Application of Vertex Distance to CC

Theorem 8 (B.–Weber)

If $\mathrm{diam}(\mathcal{G}) > \mathrm{diam}(\mathcal{H}) \geqslant D \geqslant 1$ and if $\mathcal{H}$ admits exactly two connected components, then any perfect $\mathcal{NS}$-strategy for the D-distance game collapses CC.

Theorem 9 (B.–Weber)

Let $\mathcal{G} \cong_{ns} \mathcal{H}$ such that $1 \leqslant D < \mathrm{diam}(\mathcal{G})$ and $\mathcal{H}$ is not connected. Assume "some symmetry" in a common equitable partition of $(\mathcal{G}, \mathcal{H})$. Then *there exists* a perfect strategy for the D-distance game of $(\mathcal{G}, \mathcal{H})$ that collapses CC.

(Other results are presented in the article.)

Open Question

This work raises the following question (left open):

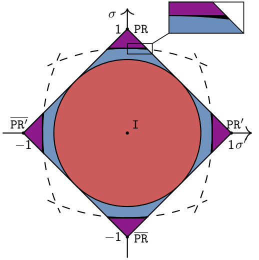
Open Question (Lovász-type)

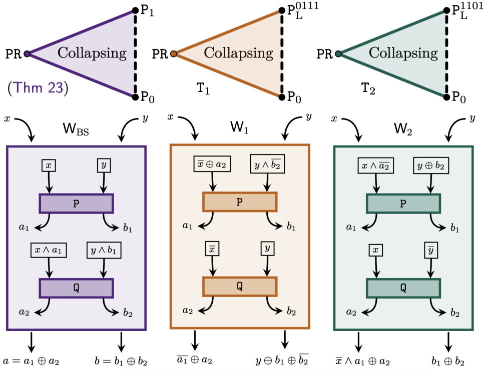
Isom.	Homomorphism countings
$\mathcal{G} \cong \mathcal{H}$	• $\forall \mathcal{K}, \# \operatorname{Hom}(\mathcal{K}, \mathcal{G}) = \# \operatorname{Hom}(\mathcal{K}, \mathcal{H})$ [Lovász'67] • $\forall \mathcal{K}, \# \operatorname{Hom}(\mathcal{G}, \mathcal{K}) = \# \operatorname{Hom}(\mathcal{H}, \mathcal{K})$ [Chaudhuri–Vardi'93]
$\mathcal{G} \cong_q \mathcal{H}$	$\forall \mathcal{K}$ planar , $\# \operatorname{Hom}(\mathcal{K}, \mathcal{G}) = \# \operatorname{Hom}(\mathcal{K}, \mathcal{H})$ [Mančinska–Roberson'20]
$\mathcal{G} \cong_{ns}^{D} \mathcal{H}$	**???**
$\mathcal{G} \cong_{ns} \mathcal{H}$	$\forall$ tree $\mathcal{K}, \# \operatorname{Hom}(\mathcal{K}, \mathcal{G}) = \# \operatorname{Hom}(\mathcal{K}, \mathcal{H})$ [Dell–Grohe–Rattan'18]

Other Results about the Collapse of CC

B.–Broadbent–Proulx, PRL:132 (7 2024) [4]. We find that boxes above a certain ellipse collapse CC, using bias amplification by majority function:

B.–Broadbent–Chhaibi–Nechita–Pellegrini, Quantum 8, 1402 (2024) [5]. We study wirings between nonlocal boxes and use them to find boxes that collapse CC:





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