— Nonlocal Games —

How to Generate a PR Box from Graph Games?

Communication Complexity of Graph Isomorphism, Coloring, and Distance Games [6]

Pierre Botteron[1], Moritz Weber[2]. [1]INRIA, ENS de Lyon (France); [2]Saarland University, Saarbrücken (Germany).

1 Observation

The **PR** box is "powerful." For instance, it is the strongest non-signaling correlation for the CHSH game [12, 9].

2 Question

Can the **PR** box be generated via pre- and post-processings of perfect strategies for other games?

3 Case of study

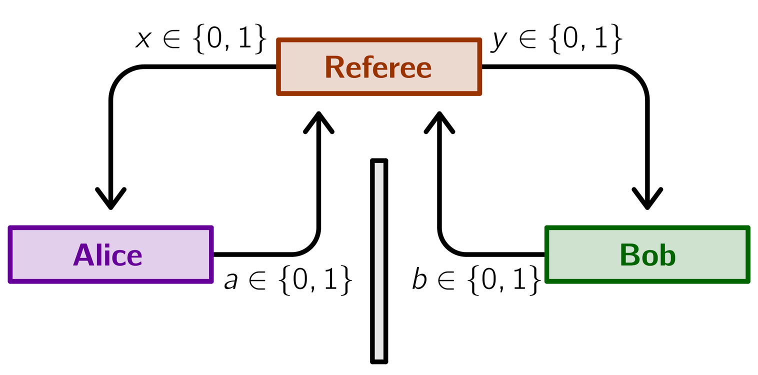
Here, we focus on some graph games.

1. Preliminary Definitions

4 Nonlocal Game

Some players Alice, Bob, Charlie, ... receive a *question* respectively $x, y, z, \ldots \in \mathcal{X}$ from a referee. Without communication, each of them outputs an *answer* respectively $a, b, c, \ldots \in \mathcal{A}$. We say that they collaboratively *win* the game if $x, y, z, \ldots$ and $a, b, c, \ldots$ satisfy a pre-defined relation, called *rule*.

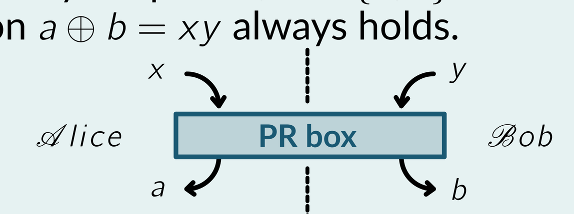
5 Example: CHSH Game [9].



Alice and Bob win at CHSH iff $a \oplus b = xy$.

6 PR Box [12]

The **PR** box is the only bi-partite non-signaling correlation with binary inputs $x, y \in \{0,1\}$ and binary outputs $a, b \in \{0,1\}$ such that the relation $a \oplus b = xy$ always holds.

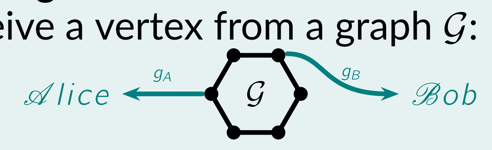


7 Other Examples of Boxes. $\mathtt{P}_{00}$ and $\mathtt{P}_{11}$ are the deterministic boxes that always give 0 (resp. 1) to Alice and Bob independently of the inputs.

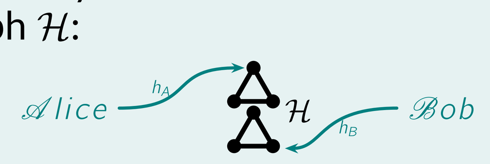
2. PR Box from the Graph Isomorphism Game

8 Graph Isom. Game [1]

Setting of the Game. Alice and Bob receive a vertex from a graph $\mathcal{G}$:



and they answer a vertex from a graph $\mathcal{H}$:



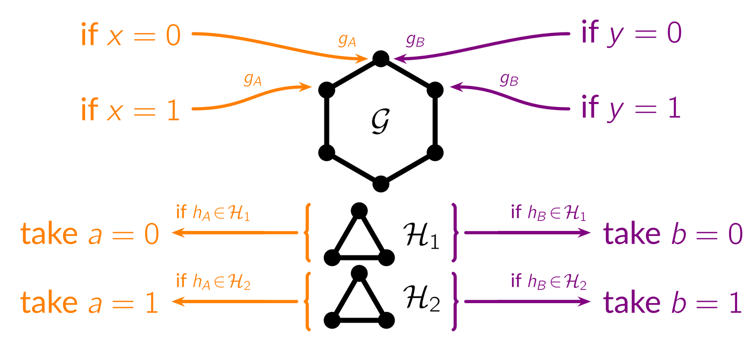
Winning Condition. Alice and Bob win the game iff:

- $g_A = g_B \Rightarrow h_A = h_B$;
- $g_A \sim g_B \Rightarrow h_A \sim h_B$;
- $g_A \not\simeq g_B \Rightarrow h_A \not\simeq h_B$.

9 Elementary Claim

If $\mathcal{G} = \mathcal{C}_6$ and $\mathcal{H} = \mathcal{C}_3 \sqcup \mathcal{C}_3$, then *any* perfect strategy for the $(\mathcal{G}, \mathcal{H})$-isomorphism game allows Alice and Bob to generate a **PR** box.

10 Proof. Let $x, y \in \{0,1\}$. We want to generate $a, b \in \{0,1\}$ such that $a \oplus b = x\,y$.



□

11 Theorem

If $\text{diam}(\mathcal{G}) \geqslant 2$ and if $\mathcal{H} = \mathcal{K}_n \sqcup \mathcal{K}_m$ where $\mathcal{K}_n, \mathcal{K}_m$ are complete graphs, then *any* perfect strategy for the $(\mathcal{G}, \mathcal{H})$-isomorphism game allows Alice and Bob to generate a **PR** box.

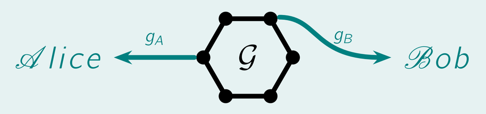
12 Theorem

Let $\mathcal{G} \cong_{ns} \mathcal{H}$ such that $\text{diam}(\mathcal{G}) \geqslant 2$ and $\mathcal{H}$ is not connected. Assume "some symmetry" in a common equitable partition of $(\mathcal{G}, \mathcal{H})$. Then *there exists* a perfect strategy for the $(\mathcal{G}, \mathcal{H})$-isomorphism game that generates a $\mathtt{PR}_{\alpha,\beta}$ box for some $\alpha > 0$, where $\mathtt{PR}_{\alpha,\beta} := \alpha\mathtt{PR} + \beta\mathtt{P}_{00} + (1 - \alpha - \beta)\mathtt{P}_{11}$.

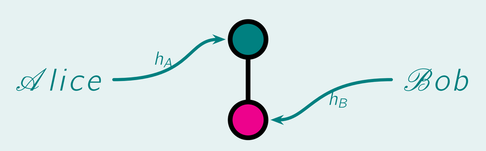
3. PR Box from the Graph Coloring Game

13 Graph Coloring Game [8]

Setting of the Game. Alice and Bob receive a vertex from a graph $\mathcal{G}$:



and they answer a color of their choice:



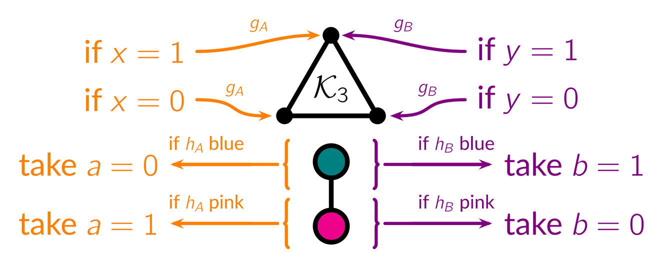
Winning Condition. They win the game if and only if:

- $g_A = g_B \Rightarrow h_A = h_B$;
- $g_A \sim g_B \Rightarrow h_A \neq h_B$.

14 Elementary Claim

A perfect strategy for the 2-coloring game of $\mathcal{K}_3$ enables to generate a **PR** box.

15 Proof. Let $x, y \in \{0,1\}$. We want to generate $a, b \in \{0,1\}$ such that $a \oplus b = x\,y$.



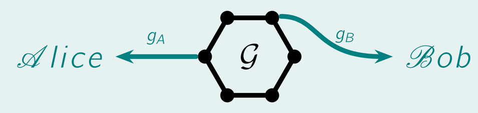
16 Theorem

Let $\mathcal{G}$ and $\mathcal{H}$ be such that $\text{diam}(\mathcal{G}) \geqslant 2$, and that $\mathcal{H}$ admits exactly N connected components $\mathcal{K}_1, \ldots, \mathcal{K}_N$, all being complete. Then, a perfect strategy for the graph isomorphism game $\mathcal{G} \cong_{ns} \mathcal{H}$, combined with a perfect strategy for the 2-coloring game of $\mathcal{K}_N$, enables to generate a **PR** box.

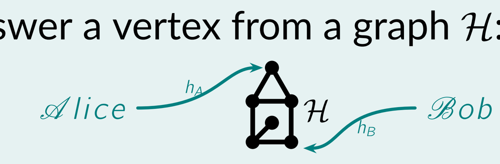
4. PR Box from the Vertex Distance Game

17 Vertex Distance Game

Setting of the Game. Alice and Bob receive a vertex from a graph $\mathcal{G}$:



and they answer a vertex from a graph $\mathcal{H}$:



Winning Condition. Let $D \in \mathbb{N}$. They win the game if and only if:

$$d(h_A, h_B) = \begin{cases} d(g_A, g_B) & \text{if } d(g_A, g_B) \leqslant D\,, \\ > D & \text{otherwise}\,. \end{cases}$$

18 Remark

The case $D = 1 \iff$ graph isomorphism game.

Denote $\mathcal{G} \cong^D \mathcal{H}$ if there exists a perfect strategy for the D-distance game of $(\mathcal{G}, \mathcal{H})$.

19 Theorem

Let $D \geqslant 2$. Then:

- $\mathcal{G} \cong^D \mathcal{H} \Leftrightarrow \mathcal{G} \cong \mathcal{H}$.
- $\mathcal{G} \cong^D_q \mathcal{H} \Leftrightarrow \mathcal{G} \cong_q \mathcal{H}$.
- $\mathcal{G} \cong^D_{ns} \mathcal{H} \not\Leftrightarrow \mathcal{G} \cong_{ns} \mathcal{H}$.

20 Theorem

If $\text{diam}(\mathcal{G}) > \text{diam}(\mathcal{H}) \geqslant D \geqslant 1$ and if $\mathcal{H}$ admits exactly two connected components, then any perfect strategy for the D-distance game generates a **PR** box.

21 Theorem

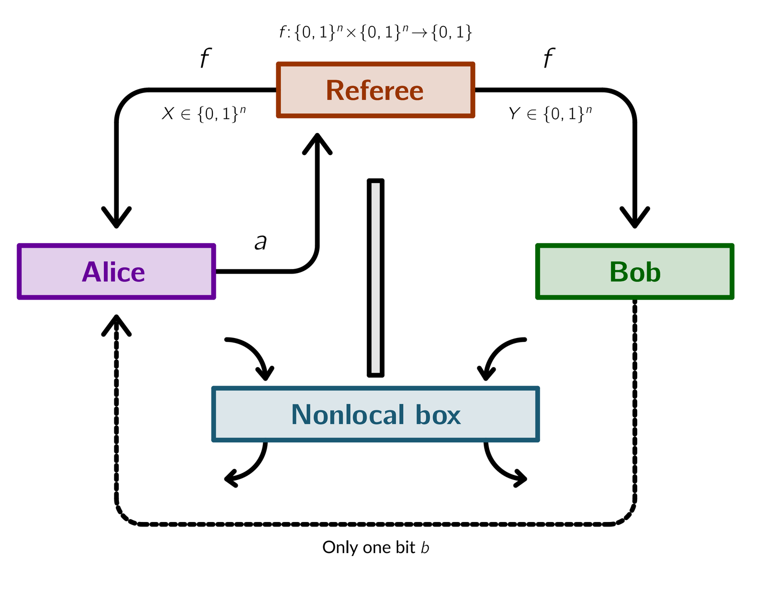
Let $\mathcal{G} \cong^D_{ns} \mathcal{H}$ for some $1 \leqslant D < \text{diam}(\mathcal{G})$ and such that $\mathcal{H}$ is not connected. Assume "some symmetry" in a common equitable partition of $(\mathcal{G}, \mathcal{H})$. Then *there exists* a perfect strategy for the D-distance game of $(\mathcal{G}, \mathcal{H})$ that generates a $\mathtt{PR}_{\alpha,\beta}$ box for some $\alpha > 0$.

5. Application to Communication Complexity

22 Definition [14, 10]

A nonlocal box $\mathtt{P}$ *collapses CC* if $\exists q > 1/2$ such that $\forall n \in \mathbb{N}, \forall f : \{0,1\}^{2n} \to \{0,1\}$, and $\forall X, Y \in \{0,1\}^n$, we have:

$$\mathbb{P}(a = f(X,Y) \,|\, X, Y, \mathtt{P}) \geqslant q\,.$$



Win $\iff a = f(X,Y)$.

23 Example

- The **PR** collapses CC [13]. Same for $\mathtt{PR}_{\alpha,\beta}$ with $\alpha > 0$ [4, 7].
- Quantum boxes cannot collapse CC [11].

24 Corollary

All the protocols from the former theorems collapse communication complexity. Hence, although non-signaling, they do not satisfy the axioms of quantum mechanics.

Future Work

25 Future Work

Investigating connections between other games and the **PR** box.

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References

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